Upward Planar Graphs and their Duals[☆,☆☆]

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Abstract

We consider upward planar drawings of directed graphs in the plane (**UP**), and on standing (**SUP**) and rolling cylinders (**RUP**). In the plane and on the standing cylinder the edge curves are monotonically increasing in y-direction. On the rolling cylinder they wind unidirectionally around the cylinder. There is a strict hierarchy of classes of upward planar graphs: **UP** $\subset$ **SUP** $\subset$ **RUP**.

In this paper, we show that rolling and standing cylinders switch roles when considering an upward planar graph and its dual. In particular, we prove that a strongly connected graph is **RUP** if and only if its dual is a **SUP** dipole. A dipole is an acyclic graph with a single source and a single sink. All **RUP** graphs are characterized in terms of their duals using generalized dipoles. Moreover, we obtain a characterization of the primals and duals of **wSUP** graphs which are upward planar graphs on the standing cylinder and allow for horizontal edge curves.

Keywords: planar graphs, dual graphs, graph drawing, upward planarity, surfaces

1. Introduction

Directed graphs are used as a model for structural relations where the vertices represent entities and the edges express dependencies. Such graphs are often acyclic and are drawn as hierarchies using the framework introduced by Sugiyama et al. [30]. This drawing style transforms the edge direction into a geometric direction: all edges point upward. If only planar drawings are allowed, we obtain *upward planar graphs*, **UP** for short. These graphs can be drawn (straight-line)

☆Supported by the Deutsche Forschungsgemeinschaft (DFG), grant Br-835/15-2

☆☆A preliminary version appeared at WG 2012 [2]

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in the plane such that the edge curves are monotonically increasing in y-direction and do not cross. Such drawings respect a unidirectional flow of information and planarity.

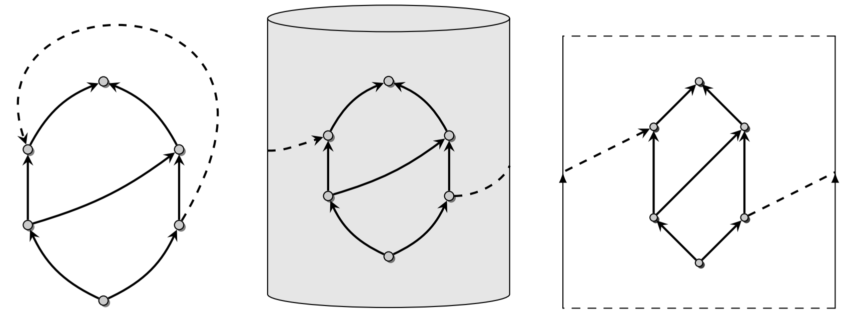
Independently, Platt [29], Kelly [25], and Di Battista and Tamassia [12] characterized the upward planar graphs as the subgraphs of planar st-graphs. An *st-graph* is a directed acyclic graph with a single source s and a single sink t and the edge (s, t). The recognition problem for **UP** is $\mathcal{NP}$-hard in general [19], and it is solvable in linear time if the graphs have a single source [24] or in polynomial time if they are 3-connected [9], outer planar [11, 18, 28], and series-parallel [11, 13].

Upward planarity on surfaces other than the plane generally deals with drawings of graphs on a fixed surface in $\mathbb{R}^3$ such that the curves of the edges are monotonically increasing in y-direction. Examples are the standing [10, 20, 26, 27, 32] and rolling cylinders [10], the sphere and the truncated sphere [15, 17, 21, 23], and the lying and standing tori [14, 16]. In full generality upward planarity is defined on arbitrary two-dimensional manifolds endowed with a vector field prescribing the direction of the edges [3, 22]. Then the standing cylinder corresponds to the plane with a radial field where the direction is away from the center, and the rolling cylinder corresponds to the plane with a concentric field where the direction is circular. Alternatively, the plane and the standing and rolling cylinders can be represented by the fundamental polygon, where the plane is identified with $I \times I$, and I is the open interval from -1 to $+1$. We obtain $I_\circ$ by identifying the boundaries of I. The *standing* (*rolling*) *cylinder* is then defined by $I_\circ \times I$ ($I \times I_\circ$), i. e., by identifying the left and right (upper and lower) boundaries of the fundamental polygon. It is well known that every undirected planar graph has a planar drawing on any surface of genus 0, such as the plane, the sphere, and the cylinder. This does no longer hold for upward planar drawings of directed graphs.

The extension of upward planarity to the sphere was addressed by Rival and his co-authors for straight-line drawings of partial orders [17, 22, 23]. Thomassen [32] considered such drawings on the standing cylinder. We call a graph **SUP** graph if it has a planar upward drawing on the standing cylinder. This is equivalent to upward planarity on the sphere where all edge curves are increasing from the south pole s to the north pole t. The equivalence was formally established in [3]. **SUP** graphs closely resemble **UP** graphs and were characterized as spanning subgraphs of planar dipoles [20, 21, 26, 32]. A *dipole* is a directed acyclic graph with a single source s and a single sink t which in contrast to an st-graph does not have to contain the edge (s, t).

If there is an st-edge, then edges cannot wind around the cylinder and the sphere, which enforces that a **SUP** graph with an st-edge is **UP**. Clearly, $\mathbf{UP} \subset \mathbf{SUP}$, where the graph from Fig. 1 shows the strictness of the inclusion [12, 25]. The recognition of **SUP** graphs is $\mathcal{NP}$-hard [23] and is solvable in linear time for 3-connected graphs with a single source [15]. Graphs in $\mathbf{SUP} - \mathbf{UP}$ need edges that wind around the cylinder, but no single edge has to make a complete winding [10]. **SUP** graphs play an important role in radial drawings [5].

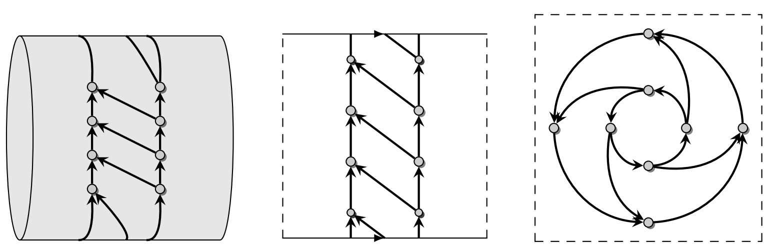
Upward planar drawings on the rolling cylinder substantially differ from



(a) An **UP** graph (without the dashed edge). (b) Its **SUP** drawing. (c) Fundamental polygon representation.

Figure 1: An example of an **SUP** graph that is not **UP**.

those in the plane and on the standing cylinder. In particular, they allow for cycles winding around the cylinder. In the fundamental polygon upward means in y-direction. Such drawings arise from recurrent hierarchies [6, 30] by the restriction to planarity. An example is shown in Figs. 2(a) and 2(b). Let **RUP** denote the set of graphs with an upward plane drawing on the rolling cylinder. Visibility representations and **RUP** drawings of planar graphs were studied by Tamassia and Tollis [31]. However, there is no characterization of **RUP** graphs alike that of **UP** and **SUP** graphs. Again, the recognition problem for **RUP** graphs is $\mathcal{NP}$-hard [10], and there is a linear-time algorithm to test whether a directed graph without sources and sinks is **RUP** [1, 4].



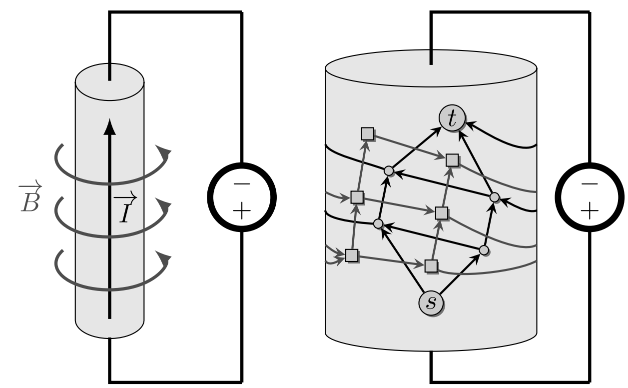
(a) Rolling cylinder. (b) Fundamental polygon. (c) Cyclic representation.

Figure 2: Different representations of a **RUP** graph.

The relaxation of monotone to non-decreasing edge curves has no effect for **UP** and **RUP** graphs, since horizontal edge segments can be avoided by a lifting [10]. However, non-decreasing curves allow for horizontal cycles around the standing cylinder or the sphere. Such drawings are called *weakly standing upward planar* and the respective class of graphs is **wSUP** [3, 10].

There are strict hierarchies **UP** $\subset$ **SUP** $\subset$ **RUP** and **SUP** $\subset$ **wSUP**, whereas **wSUP** and **RUP** are incomparable [3, 10].

This paper contains advances on the study of upward planar drawings on the standing and the rolling cylinders, and provides new characterizations for **RUP**, **SUP** and **wSUP** graphs in terms of their directed duals. One of the key results towards our characterization has a physical interpretation: Ampère's law from electromagnetism states that an electric current $\overrightarrow{I}$ flowing through a conductor generates a magnetic field $\overrightarrow{B}$ winding around the conductor (see Fig. 3(a)). In Fig. 3(b), the conductor is a planar dipole with connectors s and t in an upward planar drawing on the standing cylinder, i. e., **SUP**. The magnetic field corresponds to the dual winding around the cylinder, and is **RUP** and strongly connected.



(a) Ampère's law: Electric current flowing through a conductor generates a magnetic field that winds around the conductor.

(b) Ampère's law for dual graphs: The primal graph is a dipole with "connectors" s and t, which induces a strongly connected **RUP** graph as its dual.

Figure 3: Ampère's law.

In particular, our results are the following:

- A graph is **RUP** if and only if it has a supergraph whose dual has specific properties (Theorem 4.1). This characterization complements those already known for **UP** and **SUP** graphs as spanning subgraphs of st-graphs and dipoles, respectively.

- A graph is a strongly connected **RUP** if and only if its dual is a dipole (Theorem 4.2). This result resembles the popular Ampère's law from electromagnetism, as depicted in Fig. 3.

- A graph is **wSUP** if and only if it is has supergraphs with different properties (Theorems 5.1 and 5.2); in particular, it is shown that **wSUP** graphs always have a supergraph whose dual is **RUP**.

After some preliminaries in Section 2 we introduce tools in Section 3 to characterize **SUP** and **RUP** graphs in Section 4 and **wSUP** in Section 5. We conclude with a discussion of our findings in Section 6.

2. Preliminaries

We consider connected, planar, directed (multi-)graphs $G = (V, E)$ with non-empty sets of vertices V and edges E, where multiple edges and self-loops are allowed. A (directed) path p in G from a vertex u to a vertex v is denoted by $p = u \rightsquigarrow v$. A path p is simple if it contains no vertex twice. A cycle C is a path starting and ending at the same vertex, and C is called simple if only the start and end vertex coincide.

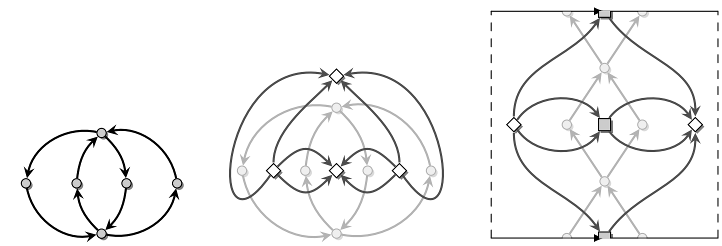
A drawing Γ of a graph G maps each vertex $v \in V$ to a *position* Γ_v in the plane and each edge $e = (u, v)$ to an *edge curve* Γ_e which is a non-self-intersecting Jordan arc between the endpoints. The *inner part* consists of all points of Γ_e except the endpoints. Γ is *planar* if all vertex positions are distinct, no vertex lies on the inner part of an edge curve, and no two edge curves share points except for common endpoints. G is *planar* if it admits a planar drawing. A planar drawing Γ partitions the plane into topologically connected regions, called *faces*, and it determines a *planar rotation system* consisting of a cyclic ordering of the edges incident to each vertex. The boundary of each *face* f is given by an undirected cycle C of vertices and edges, such that two successive edges are a successor (predecessor) at their common endpoint according to the rotation system. C is a *clockwise traversal* of f, and the edges and vertices of C are *incident* to f. An *embedding* of a graph is an equivalence class of planar drawings, where two drawings are equivalent if they have the same rotation system. An embedding of a graph is described by the set of faces or by the rotation system.

An embedding of a (*primal*) graph G defines a unique *(directed) dual* graph $G^* = (F, E^*)$, whose vertex set is the set of faces F and there is a one-to-one correspondence between the edges of G and G^*. If the counterclockwise traversal of a face f passes an edge e in its direction, we say that f is to the *left* of e. Otherwise, f is to the *right* of e. The *dual edge* e^* crosses its *primal* edge $e \in E$ from the face to the left of e to the face to the right of e, which is left to right in the direction of e. Note that G^* may be a multi-graph with self-loops. We call a vertex or an edge of an (embedded primal) graph G *rightmost* if it is incident to a sink of the dual G^*. A cycle in G is *rightmost* if it is the boundary of a sink in G^*. *Leftmost* is defined accordingly using a source of G^*.

An embedding of a primal graph G implies an embedding of its dual G^*. An embedding of a graph is an X *embedding* with $X \in \{\mathbf{RUP}, \mathbf{SUP}, \mathbf{wSUP}, \mathbf{UP}\}$ if it is obtained from an X drawing. Note that a planar graph may have an exponential number of embeddings, some of which are X embeddings and others are not.

Assumption 1. *Any X graph with $X \in \{\boldsymbol{RUP}, \boldsymbol{SUP}, \boldsymbol{wSUP}, \boldsymbol{UP}\}$ is always given with an X embedding.*

This assumption is meaningful, as otherwise it distorts the dual graph which would come from an illegal embedding, see Fig. 4.



(a) A **RUP** graph G which is not **RUP**-embedded. (b) The dual of G's embedding in Fig. 4(a). (c) A **RUP** embedding of G and its dual.

Figure 4: The duals of non-**RUP**-embedded **RUP** graphs are never dipoles.

Duality is an involution for undirected planar graphs, such that $G = G^{**}$. For directed graphs, the dual of the dual is the converse of the primal, where all edge directions are reversed. This transforms upward to downward, but it preserves the above classes of upward planar graphs.

Note that there is the convention to reverse the direction of the dual of the st-edge in an st-graph such that is crosses from right to left [12, 25], which is necessary for the fact that the dual of a planar st-graph is an st-graph. This convention induces a left and a right outer face and preserves the acyclicity of G^*.

Sources and sinks play a particular role in acyclic graphs. An acyclic graph is a *dipole* if it has exactly one source and one sink. A graph is *closed* if it has no sources and sinks. Figs. 6(a) and 6(b) show a **RUP** drawing of a closed graph and its dual.

A *dicut* of a graph G is a partition of the set of vertices into non-empty subsets X and $V \setminus X$ such that there is no edge from $V \setminus X$ to X. We refer to the edges $E_X = \{(x, y) \in E \mid x \in X \text{ and } y \in V \setminus X\}$ as *dicut set*. Dicuts and strong connectivity are complementary and the latter is dual to acyclicity [7].

Proposition 2.1. *A graph G is strongly connected if and only if G has no dicut.*

Proposition 2.2. *A graph G is acyclic (strongly connected, respectively) if and only if its dual G^* is strongly connected (acyclic, respectively).*

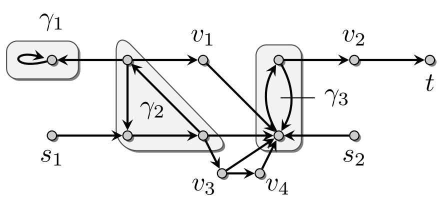
3. Compound Graphs and Pathways

Our characterization of **SUP** and **RUP** graphs is based on tools such as strongly connected components, compounds, transits, dipoles, and pathways.

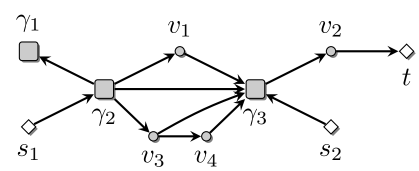
The *component graph* $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ of a graph G is an acyclic multigraph and is obtained by contracting each (strongly connected) component to a vertex and keeping all edges between the components. It inherits the embedding from G. A component is called *compound* if it contains at least one edge, which may be a self-loop. Otherwise, it is a *trivial component*.

The *terminals* $\mathbb{T} \subseteq \mathbb{V}$ are the trivial components, which are a sink or a source in both G and $\mathbb{G}$. The *compound graph* $\overline{\mathbb{G}} = (\mathbb{V}_C \cup \mathbb{T}, \overline{\mathbb{E}})$ has the compounds and terminals as its vertices and transits as its edges. A *transit* $\tau = (u, v)$ represents the collection of paths $u \rightsquigarrow v$ between two compounds or terminals u, v in the component graph $\mathbb{G}$ that intermediately visit only trivial components. All these paths are subsumed to a single edge τ in $\overline{\mathbb{G}}$. We identify each transit with the collection of paths it represents. Note that the compound graph is a simple acyclic graph.

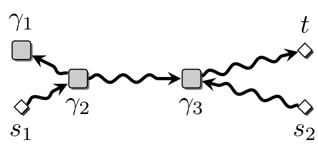
For an illustration see Fig. 5, where compounds are shaded, terminals are white diamonds, and transits are displayed as sinuous lines. For instance, there is a transit from γ_3 to t as there is the path $p = (\gamma_3, v_2, t)$ in the component graph (Fig. 5(b)) that traverses the trivial component v_2. The transit from γ_2 to γ_3 in Fig. 5(c) corresponds to the induced dipole consisting of the compounds γ_2 and γ_3 and the trivial components v_1, v_3, and v_4.



(a) A planar graph G. The compounds are enclosed by shaded regions.



(b) The component graph $\mathbb{G}$, where the compounds are displayed as rectangles and the terminals as diamonds.



(c) The compound graph $\overline{\mathbb{G}}$, where the transits are displayed as sinuous lines.

Figure 5: A graph, its component graph, and its compound graph.

Based on these definitions, we can define the key notion for our characterization.

Definition 3.1. *A graph is a* pathway *if it has a single source s and a single sink t and its compound graph is a path from s to t.*

Note that a pathway is a graph with a linear structure, where the compounds and transits are totally ordered. An acyclic pathway is a dipole and its compound graph is an edge. The graph underlying Fig. 5(c) is not a pathway, and it becomes one after removing compound γ_1 and terminal s_2. The following characterization of pathways will be particularly useful.

Lemma 3.1. *Let $G = (V, E)$ be a graph with at least one source s and at least one sink t. G is a pathway if and only if the following conditions hold:*

(i) For every vertex $v \in V$, there are paths $s \rightsquigarrow v$ and $v \rightsquigarrow t$.

(ii) Every path $s \rightsquigarrow t$ contains at least one vertex of each compound.

Proof. "⇒": Follows from the definition of a pathway.

"⇐": Let s be a source and t be a sink of G. Since $s \rightsquigarrow v$ and $v \rightsquigarrow t$ for every $v \in V$, s is the single source and t the single sink of G. Let p be a path from s to t in G. Then p contains at least one vertex from each compound. Whenever p leaves a compound γ it does so by an edge $e = (u, v)$ that belongs to a transit. In particular, p can never return to γ as otherwise v would also belong to γ. Hence, p corresponds to a Hamiltonian path $\overline{p} = s \rightsquigarrow t$ in $\overline{\mathbb{G}}$. Together with the fact that $\overline{\mathbb{G}}$ is acyclic this implies that $\overline{\mathbb{G}}$ is a path from s to t. □

4. RUP Graphs and their Duals

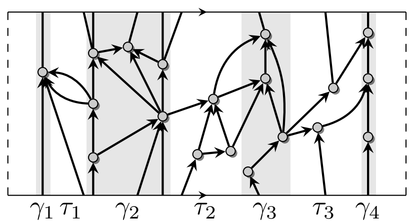
UP and **SUP** graphs have been characterized as spanning subgraphs of st-graphs and dipoles, respectively. We complement these results by a dual characterization of **RUP** graphs in Theorems 4.1 and 4.2, which consider subgraphs of closed graphs and strongly connected graphs, respectively, and which both use Lemma 4.1. Lemma 4.2 completes the proof of Theorem 4.1 by the construction of a **RUP** drawing.

Theorem 4.1. *A graph is* ***RUP*** *if and only if it is a spanning subgraph of a closed planar graph whose dual is a pathway.*

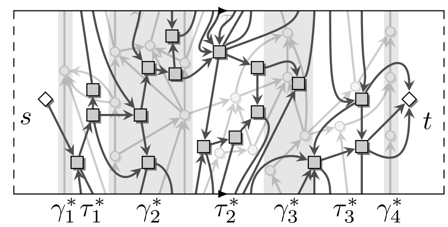
For an illustration, consider the **RUP** drawing of graph G in Fig. 6(a), where compounds are drawn on a shaded background. The component graph $\mathbb{G}$ of G is displayed in Fig. 6(c) along with its compound graph $\overline{\mathbb{G}}$ below. Consider compound γ_2 and its dual displayed in Fig. 7(a), and transit τ_2 and its dual in Fig. 7(b). As γ_2 is strongly connected, its dual is acyclic. In other words, compounds and transits switch roles from primal to dual. The path-like structure of compound graph $\overline{\mathbb{G}}$ is reflected in the compound graph of the dual G^*, which we denote by $\overline{\mathbb{G}}^*$ (cf. 6(d)). Furthermore, all cycles in the **RUP** drawing have the same orientation and wind around the cylinder. Therefore, all transits of G^* point into the same direction. Finally, G contains neither sources nor sinks and, thereby, there is a leftmost and a rightmost cycle at the left and right borders of G's **RUP** drawing. In the dual, the leftmost cycle encloses a source s and the rightmost cycle a sink t. In fact, s is the single source and t the single sink of G^*. These observations indicate that the compound graph of G^* is a path $s \rightsquigarrow t$ and G^* is a pathway.

Lemma 4.1. *The dual of a closed* ***RUP*** *graph is a pathway.*

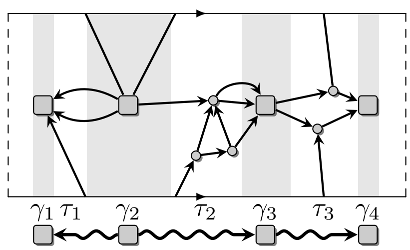
Proof. Consider a closed graph G with a **RUP** embedding. Then its dual $G^* = (F, E^*)$ has at least one source and at least one sink. Let $f \in F$ be the



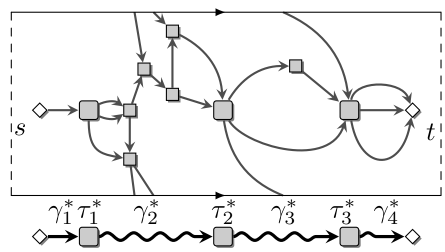
(a) **RUP**-embedded graph G.



(b) Dual G^* of graph G.

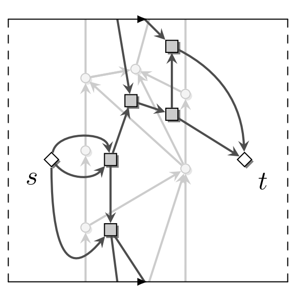


(c) The component graph $\mathbb{G}$ and the compound graph $\overline{\mathbb{G}}$ of G.

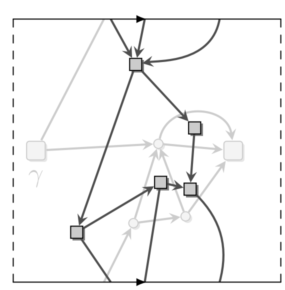


(d) The component graph $\mathbb{G}^*$ and the compound graph $\overline{\mathbb{G}}^*$ of the dual G^* with $s, t \in \mathbb{T}$.

Figure 6: A **RUP**-embedded graph with its dual and their component and compound graphs.



(a) The dual of compound γ_2 of G is a transit of G^*.

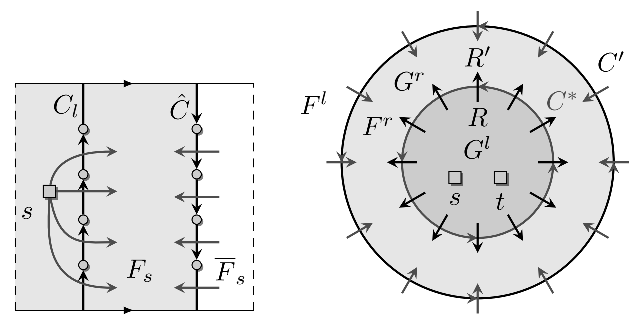


(b) The dual of transit τ_2 of G is a compound of G^*.

Figure 7: Transits and compounds switch roles from primal to dual.

leftmost face of G, which corresponds to the region between the left boundary of the cylinder and edge curves of G's drawing. As G is closed, it contains at least one cycle that winds exactly once around the cylinder [10], and separates the left and right boundaries of the cylinder. Hence, the part of f's boundary that belongs to G's drawing winds exactly once around the cylinder. Let v be any vertex incident to f. As G is closed and **RUP**-embedded, v has one incoming and one outgoing edge that are both incident to f. Hence, f's boundary is a cycle and f is a source in G^*. By an analogous reasoning, there is also a sink in G^* that is the rightmost face.

Let s be a source and t be a sink in G^*. We use Lemma 3.1 to show that G^* is a pathway. Let F_s be the set of faces reachable from s with $F_s = \{f \in F \mid s \rightsquigarrow f \text{ in } G^*\}$. If $F_s \subsetneq F$, then F is partitioned into F_s and $\overline{F}_s = F \setminus F_s$ and $(\overline{F}_s, F_s)$ is a dicut of G^*. Let E^*_s be the corresponding dicut set. The primal edges of E^*_s, denoted by E_s, form a cycle $\hat{C}$ in G. Let C_l be the leftmost cycle in G that encloses source s. Fig. 8(a) illustrates the situation, where the shaded area covers the faces F_s. Then $\hat{C}$ winds around the cylinder in the opposite direction of C_l, which contradicts our assumption of a **RUP** embedding. Hence, every face f of G^* is reachable from s. Accordingly, there is a path from every face to sink t. Therefore, s is the single source and t the single sink of G^*.



(a) Two cycles with opposite orientations. (b) s and t are within region R.

Figure 8: Situation obtained in the proof of Lemma 4.1.

It remains to show that every $p^* = s \rightsquigarrow t$ in G^* contains a face of each compound in G^*. First, if G is strongly connected, then G^* is acyclic and contains no compounds at all, and we are done. If G^* contains at least one compound γ^* with a cycle C^*, then p^* contains at least one face of C^* and, therefore, a face of γ^*. For contradiction, suppose that p^* contains no face of C^*. Consider a planar drawing of G^* that respects the embedding. In this drawing C^* encloses a region R. If s and t are at opposite sides of C^*, then p^* must cross an edge of C^* to connect s and t by Jordan's curve theorem, contradicting planarity. Hence, s and t are either both inside or outside R. First, we assume the former and depict the situation in Fig. 8(b). In the primal graph G, C^* defines a dicut (V^l, V^r) with $V^r = V \setminus V^l$. Let $G^r = (V^r, E^r)$ be the subgraph of

G induced by V^r. Depending on the orientation of C^*, G^r lies either completely outside or inside of R. Suppose that G^r lies outside; the inside case is analogous. Each vertex of G^r has at least one outgoing edge as G is closed and there is no edge that points from a vertex in V^r to a vertex in V^l. In other words, G^r contains no sink and, hence, it contains a cycle C', where C' encloses a region R' such that $R \subsetneq R'$ (as in Fig. 8(b)) or $R' \subsetneq R$. Since both s and t lie within R, cycle C' defines a dicut (F^l, F^r) in G^*, where either $s, t \in F^l$ or $s, t \in F^r$ depending on the orientation of C' and on whether $R \subsetneq R'$ or $R' \subsetneq R$. If $s, t \in F^l$, there is no path from any face in F^r to t, and if $s, t \in F^r$ there is no path from s to any face in F^l (see Fig. 8(b)); a contradiction to (i) in Lemma 3.1. The case where both s and t are outside of R is analogous. Then p^* contains a vertex from C and, thus, a vertex from γ^* which implies (ii) in Lemma 3.1. □

Fig. 3 illustrates the relationship between Ampère's law from physics and Ampère's law for dual graphs which states that **SUP** and **RUP** switch roles from primal to dual. This relationship is expressed by the following theorem.

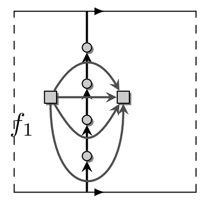
Theorem 4.2 (Ampère's Law for Duals)**.** *A graph is strongly connected an **RUP** if and only if its dual is a dipole.*

Proof. A strongly connected graph G is closed such that its dual is acyclic by Proposition 2.2 and a dipole if G is **RUP** by Lemma 4.1.

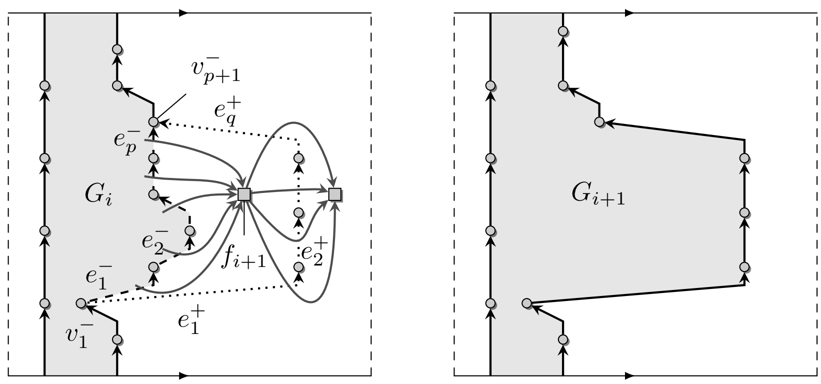
For the converse direction, we inductively construct a **RUP** drawing of G on the fundamental polygon of the rolling cylinder such that the embedding of G is preserved. As G's dual G^* is a dipole, we obtain a topological ordering $f_1, \dots, f_k$ $(k \leq 1)$ of the faces, where f_1 is the single source and f_k is the single sink of G^*. Let G_i $(1 \leq i \leq k)$ be the embedded subgraph of G induced by the faces $f_1, \dots, f_i$, i. e., G_i contains exactly those edges and vertices bounding the faces $f_1, \dots, f_i$.

The idea is to add edges to G_i such that f_{i+1} is enclosed as a new face and f_{i+1} lies to the left of all newly added edges. To assure a planar drawing, the x-coordinates of the newly added vertices are strictly greater than the x-coordinates of all vertices in G_i. Let $x_1, \dots, x_k$ with $x_i \in I$ for $1 \leq i \leq k$ be a sequence of strictly increasing x-coordinates, i. e., $-1 < x_i < x_{i+1} < 1$ for all $1 \leq i < k$. As induction invariant, for each G_i we obtain a **RUP** drawing Γ_i, which respects the embedding of G_i and lies within $[x_1, x_i] \times I_\circ$. Additionally, the dual G_i^* of each G_i is a planar dipole. Especially, the right boundary of Γ_i is a directed cycle and all faces $f_1, \dots, f_i$ are to the left of this cycle. The construction of the drawing must assure that the next face can always be added without causing a crossing.

For the base case, consider G_1. Since f_1 is a source in G^*, G_1 consists of a single cycle C with $\mathrm{d}^+(f_1)$ many edges, where $\mathrm{d}^+(f_1)$ is the outdegree of f_1. All vertices of C get the x-coordinate x_1 and the y-coordinates are chosen according to the cyclic order as defined by C, where the total order induced by the y-coordinates of the vertices implies the cyclic order as defined by C. See Fig. 9(a) for an illustration. The drawing of G_1 guarantees the induction invariants. Especially, G_1^* is a dipole with source f_1 and a single sink to the right of C.



(a) Base case.



(b) Induction step.

(c) After the induction step.

Figure 9: Inductive construction of a **RUP** drawing from its dual.

For $1 < i < k-1$ the situation is depicted in Fig. 9(b). In the embedding of G_{i+1}^*, all incoming edges are consecutive in the rotation system of f_{i+1} and so are all outgoing edges, i. e., the rotation system of f_{i+1} is bimodal [8, 31]. This follows from the fact that G^* is **SUP**-embedded as an embedded planar dipole [21]. Denote by $e_1^-, \ldots, e_p^-$ (dashed) and $e_1^+, \ldots, e_q^+$ (dotted) the primal edges of all incoming and outgoing edges of f_{i+1}, respectively, where the sequence of duals of $e_1^+, \ldots, e_q^+, e_p^-, e_{p-1}^- \ldots, e_1^-$ in order is the rotation system of f_{i+1}. Note that f_{i+1} has at least one incoming edge as otherwise it would be a source different from f_1. Analogously, f_{i+1} has at least one outgoing edge. Due to the topological ordering of the faces, all faces with an outgoing edge to f_{i+1} are in the drawing of G_i. Additionally, all edges e_j^- are part of the rightmost cycle C_r of G_i. For contradiction assume that is not the case. Then, there is an edge e_j^- $(1 \leq j \leq p)$ where the endpoints of its dual are both to the left of C_r. However, then either e_j^- cannot be an edge bounding f_{i+1} or Γ_i does not respect the embedding of G_i; a contradiction to the induction hypothesis. Moreover, since Γ_i respects the rotation system of f_{i+1}, the edges $e_1^-, \ldots, e_p^-$ form a path $p^- = (v_1^-, \ldots, v_{p+1}^-)$ in G_i, which is part of C_r. As an illustration, path p^- is dashed in Fig. 9(b). Let $a \in I_\circ$ and $b \in I_\circ$ be the y-coordinates of v_1^- and v_{p+1}^-, respectively. Note that v_{p+1}^- must lie "above" v_1^- as p^- must be drawn upward. W. l. o. g., we assume that $a < b$, otherwise we rotate the drawing around the cylinder. Accordingly, the edges $e_1^+, \dots, e_q^+$ correspond to a path $p^+ = (v_1^+, \dots, v_{q+1}^+)$ in G_{i+1}. Note that $v_1^+ = v_1^-$ and $v_{q+1}^+ = v_{p+1}^-$ since p^+ and p^- together bound face f_{i+1}. We assign to each vertex $v_2^+, \dots, v_q^+$ the x-coordinate x_{i+1}. For every vertex v_j^+ we choose as y-coordinate a value y_j^+ such that for all $1 \leq j < q$ the inequality $a < y_j^+ < y_{j+1}^+ < b$ holds. Now, the edges of p^+ can be drawn upward as polylines with a single bend at the first and the last edge of the path; see Figs. 9(b) and 9(c). For the position of the bend in edges e_1^+ and e_q^+, we choose as y-coordinate some value in the intervals (a, y_2^+) and (y_q^+, b), respectively, such that the straight lines from v_1^- and v_{p+1}^- to the bends cause no crossing with any edge from p^-. In consequence, both edges consist of two segments of which one is vertical and the other almost horizontal. For the x-coordinate of the bends, we choose x_{i+1}. If p^+ consists of a single edge, this edge has two bends. The resulting drawing Γ_{i+1} of G_{i+1} is a **RUP** drawing respecting the embedding of G and it lies within $[x_1, x_{i+1}] \times I_\circ$. In Γ_{i+1} there is a newly formed cycle C_r' containing p^+ on the right border of the drawing such that all faces $f_1, \dots, f_{i+1}$ lie to the left of C_r', see Fig. 9(c). Thus, the dual G_{i+1}^* is again a planar dipole.

In the drawing of G_{k-1}, face f_k is already existent as it is the single face to the right of the rightmost cycle in G_{k-1}. Hence, $G_{k-1} = G_k = G$ is **RUP**-embedded. □

Since a planar dipole is **SUP** [21], Theorem 4.2 implies:

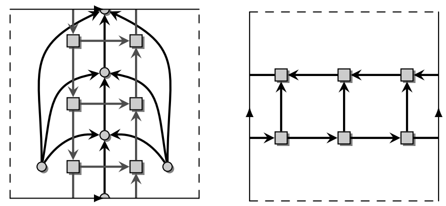
Corollary 4.1. *The dual of a strongly connected **RUP** graph is **SUP**.*

There is a duality for **SUP** and **RUP** graphs and also for graph properties such as acyclicity and strong connectivity and compounds and transits. The latter is illustrated in Figs. 6 and 7. In the dual G^* of G, compounds and transits of G switch roles. For this, consider compound γ_2 in Fig. 6(a). Compound γ_2 in Fig. 6(a) is **RUP**-embedded and its dual, depicted in Fig. 7(a), is a **SUP**-embedded transit. Similarly, the dual of transit τ_2 in Fig. 6(a) is the compound shown in Fig. 7(b). The following corollary subsumes these observations.

Corollary 4.2. *Let G be a closed **RUP** graph and let $\overline{\mathbb{G}} = (\mathbb{V}_C, \overline{\mathbb{E}})$ be its compound graph. Then, the dual of each compound in $\mathbb{V}_C$ is a **SUP**-embedded transit and the dual of each transit in $\overline{\mathbb{E}}$ is a **RUP**-embedded compound.*

Proof. Since the compound of a **RUP**-embedded graph is also **RUP**-embedded it is a **SUP**-embedded transit by Corollary 4.1. A transit of a graph is a dipole. Since G is (**RUP**-)embedded, the transit is also **SUP**-embedded [21]. Thereby, its dual is a **RUP**-embedded compound (Theorem 4.2). □

Note that several transits may induce cycles in opposite direction in their dual. The orientation of cycles in the dual is determined by the direction of the primal. For instance, the dual of τ_1 in Fig. 6(b) winds around the cylinder in upward direction as τ_1 points from γ_2 to γ_1, whereas the dual of τ_3 winds around the cylinder downwards as τ_3 points from γ_3 to γ_4. In particular, the dual of a **RUP** graph is not necessarily **RUP** [10], as Fig. 10(a) shows.



(a) A **RUP** graph G whose dual is not **RUP** (but **wSUP**).

(b) **wSUP** drawing of the dual of G.

Figure 10: The dual of a **RUP** graph is not necessarily **RUP**.

Corollary 4.3. *There are* ***RUP*** *graphs whose duals are not* ***RUP****.*

Let $\overline{\mathbb{G}}^*$ be the compound graph of the dual G^* of a closed **RUP**-embedded graph G. We denote the dual of a compound γ of G by γ^*, where γ^* is a transit of G^*. Likewise, we denote the dual of a transit τ of G by τ^*, where τ^* is a compound of G^*. By Lemma 4.1, the dual of a closed **RUP**-embedded graph is a pathway. We now prove the converse.

The basic idea is as follows: Consider the example in Fig. 6(a) and the compound graph $\overline{\mathbb{G}}^*$ of its dual G^* in Fig. 6(d). Since G^* is a pathway, $\overline{\mathbb{G}}^*$ is a path $p = (s, \gamma_1^*, \tau_1^*, \gamma_2^*, \tau_2^*, \gamma_3^*, \tau_3^*, \gamma_4^*, t)$, where γ_i^* is the dual of compound γ_i, and τ_i^* is the dual of transit τ_i. Each element on p corresponds to a subgraph in the primal G and in its component graph, i. e., for each τ_i^* there is a transit τ_i in G and for each γ_j^* there is a compound γ_j in G. We construct a **RUP** drawing of G by subsequently attaching the elements of p to each other. We start with transit γ_1^*, which is a dipole, and obtain a **RUP** drawing of its primal γ_1 which respects the given embedding by Theorem 4.2. Then, we proceed with τ_1^*, a compound in G^*, for which we obtain a **SUP** drawing of its primal τ_1 which also respects the given embedding. However, note that this **SUP** drawing is upward only on the standing cylinder. In particular, rotating the **SUP** drawing by 90 degrees to obtain a drawing on the rolling cylinder does not necessarily produce a **RUP** drawing. Fortunately, we can transform the **SUP** drawing of τ_1, while preserving its embedding, such that it is also upward on the rolling cylinder. The so obtained drawing of τ_1 is then attached to the rightmost cycle of γ_1. Then, the **RUP** drawing of γ_2 is attached to the right side of τ_1, and so forth until we reach t. Since all transits γ_j^* point into the same direction in $\overline{\mathbb{G}}^*$, all cycles of the compounds in G have the same orientation in the obtained drawing and they all wind around the cylinder in the same direction.

Lemma 4.2. *A closed graph is* ***RUP*** *if its dual is a pathway.*

Proof. Let $G = (V, E)$ be an embedded closed graph whose dual $G^* = (F, E^*)$ is a pathway. If G consists of a single compound, then it is strongly connected, G^* is a dipole, and the embedding of G is a **RUP** embedding according to Theorem 4.2.

Suppose that G contains at least two compounds. Let $s \in F$ and $t \in F$ be the source and the sink of G^*, respectively. Then each cycle C in G separates s and t, i. e., s lies to the left and t to the right of C or vice versa. C defines a dicut (F^l, F^r) in G^*. Since G^* is a pathway, there is a path from s to any face $f \in F$ and from f to t. Hence, $s \in F^l$ and $t \in F^r$, and s is to the left and t to the right of C. Let C_1 and C_2 be two cycles in different compounds of G. We show that C_1 and C_2 have the same orientation. As C_1 and C_2 belong to different compounds, they are vertex- and edge-disjoint. In the dual, C_1 and C_2 define two dicuts (F_1^l, F_1^r) and (F_2^l, F_2^r), respectively, with $s \in F_1^l, F_2^l$ and $t \in F_1^r, F_2^r$. If C_1 and C_2 have opposite orientations, we obtain the same situation as in the proof of Lemma 4.1 and as displayed in Fig. 8(a), where t is situated within region $\overline{F}_s$. In particular, there would be no path from s to t in G^* which contradicts Lemma 3.1.

By the reasoning in the previous paragraph, we can conclude that there is a total order $\gamma_1, \gamma_2, \ldots, \gamma_k$ of the compounds $\mathbb{V}_C$ of G with the following properties: The region to the left of any cycle in compound γ_i $(1 < i < k)$ contains all vertices of compounds $\gamma_1, \ldots, \gamma_{i-1}$, and the region to the right of any cycle in compound γ_i contains all vertices of compounds $\gamma_{i+1}, \ldots, \gamma_k$. Compound γ_1 is the leftmost compound in the sense that no compound is to its left side and all other compounds are to its right side. In the same sense, γ_k is the rightmost compound.

In the following, consider a drawing of G in the plane which respects the given embedding. Fig. 11 shows the structure of such a drawing. The compounds are displayed as shaded rings and the arrows on the rings' borders indicate the direction of the compounds' cycles. Face s is situated in the middle and lies to the left of all compounds $\gamma_1, \ldots, \gamma_k$. Face t is the outer face to the right of all compounds. Let γ_i and γ_j be two compounds of G with $1 \le i < j \le k$ such that $j - i > 1$, i. e., in the ordering of the compounds, there is at least one compound between γ_i and γ_j. We now show that there is no transit between γ_i and γ_j, i. e., neither $(\gamma_i, \gamma_j) \in \overline{\mathbb{E}}$ nor $(\gamma_j, \gamma_i) \in \overline{\mathbb{E}}$. Assume for contradiction that there is transit $\hat{\tau} = (\gamma_i, \gamma_j) \in \overline{\mathbb{E}}$. The opposite direction is analogous. Then there is a path p from a vertex in γ_i to a vertex in γ_j which internally visits only trivial components. Further, there is at least one compound γ_ℓ between γ_i and γ_j $(i < \ell < j)$. In other words, $\hat{\tau}$ must "overleap" γ_ℓ. Compound γ_ℓ contains at least one cycle that encloses a region R such that γ_i is completely contained within R in the drawing. As p internally only visits trivial components, p cannot have a vertex in common with γ_ℓ. Moreover, p starts within region R and must reach a vertex of γ_j which is situated completely outside of R. This inevitably leads to a crossing which contradicts planarity. For instance, in Fig. 11, the transit $\hat{\tau}$ points from γ_1 to γ_3 and γ_2 is situated between them, which leads to a crossing.

Since (by assumption) G is connected, there is a transit between adjacent compounds γ_i and γ_{i+1}, i. e., for all i with $1 \leq i < k$, either $(\gamma_i, \gamma_{i+1}) \in \overline{\mathbb{E}}$ or $(\gamma_{i+1}, \gamma_i) \in \overline{\mathbb{E}}$. In the following, let $\gamma_1, \tau_1, \gamma_2, \ldots, \tau_{k-1}, \gamma_k$ be the sequence of compounds and transits in G such that τ_i is the transit connecting compounds γ_i and γ_{i+1}. Analogously, let $\gamma_1^*, \tau_1^*, \gamma_2^*, \ldots, \tau_{k-1}^*, \gamma_k^*$ be the sequence of compounds and transits in G^* in order of the path from the source to the sink in $\overline{\mathbb{G}}^*$.

By Theorem 4.2, each compound in G has a **RUP** embedding. Further, each transit of G is a planar dipole whose embedding is **SUP**. A **SUP** embedding is also a **RUP** embedding, which was proved in [3] and in [10] using different techniques. Hence, the embedding of each transit of G is also **RUP**. We conclude our proof by showing that the **RUP** embeddings of the individual compounds and transits can be merged into a single consistent **RUP** embedding of the whole graph G. For this, we construct a **RUP** drawing of G by subsequently processing the elements in the order $\gamma_1, \tau_1, \gamma_2, \tau_2, \ldots, \gamma_k$. For an example, see Fig. 12.

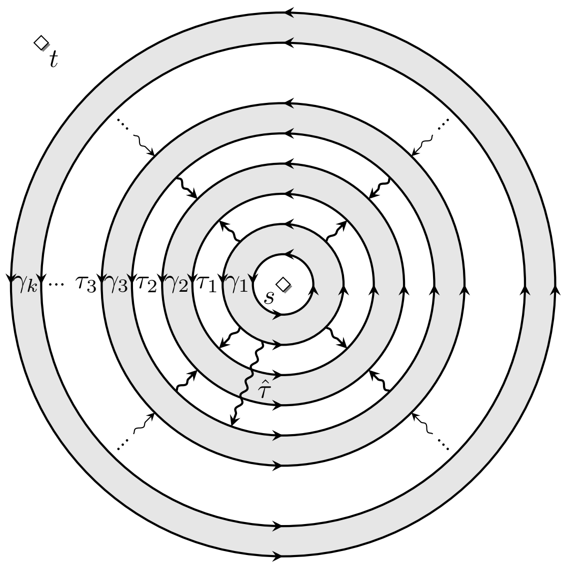
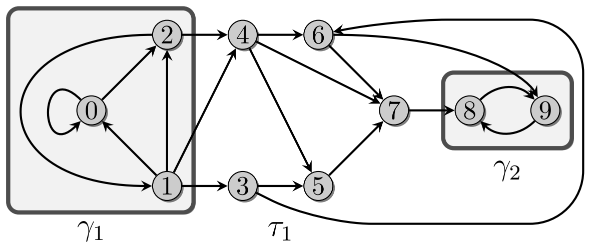
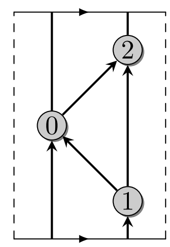


Figure 11: A transit which "overleaps" a compound causes a crossing.

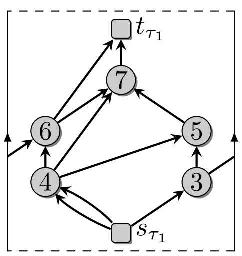
We start with transit γ_1^* of G^*, which is a dipole, and obtain a **RUP** embedding of γ_1 by Theorem 4.2 (see Fig. 12(b)). Let Γ_{γ_1} be a **RUP** drawing of γ_1 according to its **RUP** embedding. Denote by C_1^r the rightmost cycle of γ_1. We proceed with τ_1^*, a compound in G^*. The primal τ_1 is a transit and, thus, a dipole and its embedding is **SUP**. Denote by Γ_{τ_1} the **SUP** drawing of τ_1. First assume that τ_1 points from γ_1 to γ_2. As shown in [3, 10], we shear Γ_{τ_1} such that it becomes **RUP** where its source is on the left and its sink on the right side. We place the resulting drawing to the right of Γ_{γ_1} (see Fig. 12(d)). Recall that τ_1 is not a subgraph of G but of its component graph: τ_1 is a dipole, where source s_{τ_1} and sink t_{τ_1} correspond to γ_1 and γ_2, respectively. All other vertices

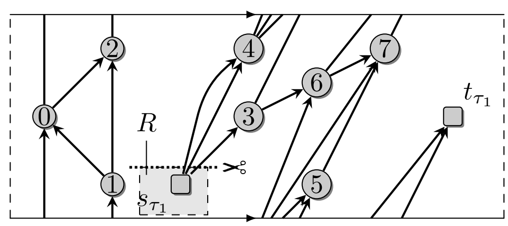




(a) A closed **RUP** graph. The subgraphs within the shaded regions are the two compounds γ_1 and γ_2. Transit τ_1 points from γ_1 to γ_2.

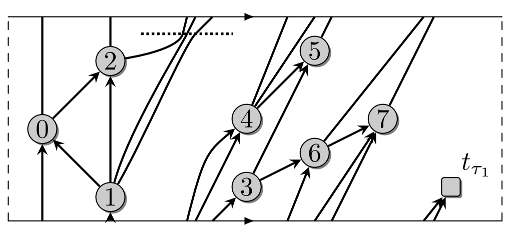
(b) **RUP** embedding of γ_1.





(c) **SUP** embedding of τ_1.

(d) Γ_{τ_1} has been sheared to become a **RUP** drawing and placed to the right of Γ_{γ_1}.



(e) Intermediate result Γ' after the drawings of γ_1 and τ_1 have been merged.

Figure 12: Situations obtained in the proof of Lemma 4.2.

of τ_1 are trivial components and directly correspond to vertices of G. The edges incident to s_{τ_1} in τ_1 correspond to edges in G which are incident to vertices in γ_1, more precisely vertices of C_1^r. Source s_{τ_1} is expanded to C_1^r and its incident edges must be drawn upward planar. We apply a compression and rotation technique similar to the one in [10]. Therefore, we remove s_{τ_1} and all points of its incident edge curves from Γ_{τ_1} within an axis-aligned rectangular region R around s_{τ_1} that is chosen as follows (see Fig. 12(d)): R contains no points of Γ_{τ_1} besides those of s_{τ_1} and its incident edges, and its dimensions are such that all intersection points of R's boundary with Γ_{τ_1} are at the top side of the rectangle. Note that such a rectangle exists as Γ_{τ_1} is upward in y-direction. This results in edge curves starting in cutting points rather than in s_{τ_1}, where all cutting points have the same y-coordinate. We rotate Γ_{τ_1} around the rolling cylinder such that the y-coordinate of the cutting points is greater than the y-coordinates of any of the vertices in γ_1. Let $e = (u, v)$ be the edge in G corresponding to the edge in τ_1 whose edge curve has the cutting point with the smallest x-coordinate. Vertex u is part of C_1^r and v is a vertex in τ_1. Next we rotate the drawing of γ_1 such that u is the topmost vertex, but has a smaller y-coordinate than the cutting points. Since both the embedding of C_1^r implied by Γ_{γ_1} and the embedding of τ_1 implied by Γ_{τ_1} obey the initial planar embedding of G, the order of the cutting points from right to left corresponds to the order of the vertices in C_1^r from bottom to top. Hence, we can connect the vertices of C_1^r with edge curves increasing monotonically in y-direction to the respective cutting points without introducing crossings.

The resulting drawing Γ' (Fig. 12(e)) forms the basis for the next step, where as in Theorem 4.2 we obtain a **RUP** drawing Γ_{γ_2} of compound γ_2 and place it to the right of Γ'. In a similar way, we remove t_{τ_1} from the drawing and reconnect the resulting cutting points to the respective vertices in the leftmost cycle of γ_2. If, contrary to our aforementioned assumption, a transit is directed from right to left, we proceed similarly except that we switch the roles of s_{τ_1} and t_{τ_1} and rotate the cutting points around t_{τ_1} to the bottom rather than the top. Analogously, we proceed with $\tau_2, \gamma_3, \tau_3, \gamma_4, \ldots$ until we have processed all components, resulting in a **RUP** drawing of G. □

Embedded **UP** and **SUP** graphs can be augmented by new edges, such that only a single source and a single sink remains. An internal source (sink) is attached by an incoming edge from a vertex in the face below. Accordingly, **RUP** graphs can be augmented to closed graphs, i. e., all sources and sinks are eliminated while preserving **RUP**.

Lemma 4.3. *A **RUP** graph is a spanning subgraph of a closed **RUP** graph.*

Proof. We iteratively add edges until all vertices have both incoming and outgoing edges. Let t be a sink of G. Shoot a ray from the position of t in upward direction and determine where it first meets some point p of the drawing. If p belongs to a vertex v, we add the edge (t, v) on the ray. Note that $v = t$ if no other vertex or edge has a point with the x-coordinate of p such that the ray winds exactly once around the cylinder. If p belongs to an edge $e = (u, v)$ follow the edge curve Γ_e

to its endpoint v, and insert the edge (t, v) in the **RUP** drawing, such that it first follows the ray, then Γ_e at a small distance, and finally meets v. Incoming edges to the sources of G are treated similarly. □

The proof of Theorem 4.1 is now complete. The only-if direction follows from Lemmas 4.1 and 4.3 and the if direction is a consequence of Lemma 4.2 and the fact that every subgraph of a **RUP** graph is a **RUP** graph.

5. wSUP Graphs and their Duals

In this section, we extend **SUP** to **wSUP** graphs and allow horizontal edge curves. Examples of **wSUP**-embedded graphs are shown in Figs. 10(b) and 13(a). We derive a characterization of **wSUP** graphs by means of compound and dual graphs.

Obviously, a **wSUP** graph has a cycle if and only if the edge curves form a horizontal band around the standing cylinder. Hence, two such cycles never interfere, but they may have opposite directions, as Figs. 10(b) and 13(a) show.

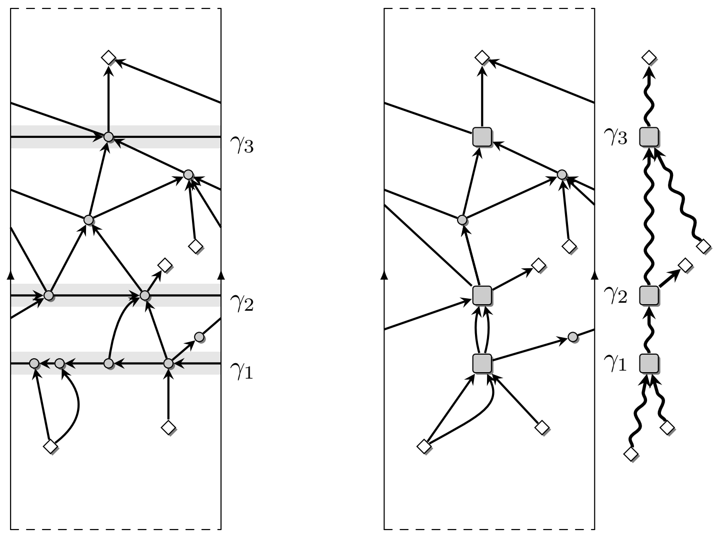
Proposition 5.1. *All cycles in a **wSUP** graph are disjoint.*

Corollary 5.1. *Each compound in a **wSUP** graph is a simple cycle.*

To characterize **wSUP** graphs, we proceed as before for **RUP** graphs in Section 4 and first expand a **wSUP** graph to a pathway. Note that the supergraph is not spanning since a new source (sink) must be added if there is a horizontal cycle at the lower (upper) end. For our proof, we extend techniques for **SUP** graphs from [21].

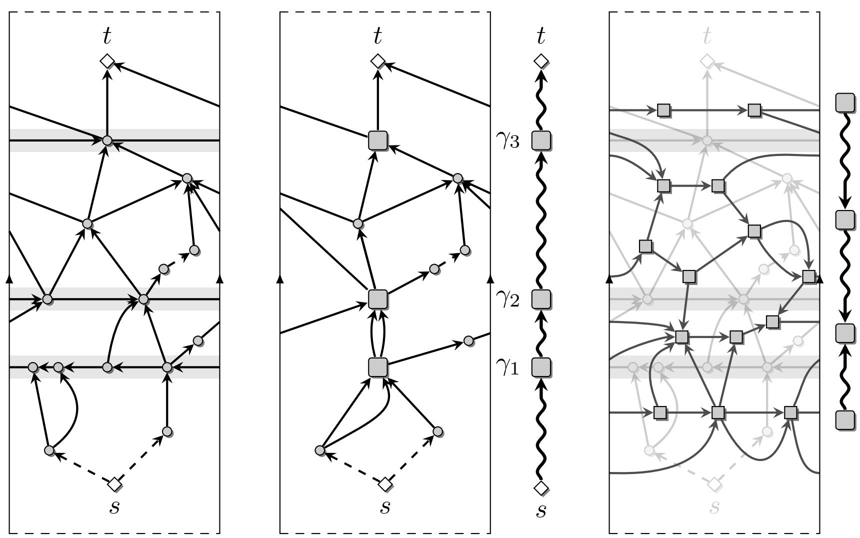
Theorem 5.1. *A graph is **wSUP** if and only if it is a subgraph of a planar pathway whose compounds are simple cycles.*

Proof. "⇒": If a graph G is acyclic, it is **SUP** [10] and thus a subgraph of a planar dipole H, whose pathway consists of a single edge (s, t) and has no compounds. Otherwise, G is composed of pairwise disjoint cycles $C_1, \ldots, C_k$, which are the non-trivial components of the component graph $\mathbb{G}$, and of acyclic subgraphs $S_0, \ldots, S_{k+1}$. S_0 is below C_1 and some of its edges end at vertices of C_1. If S_0 is empty, we extend it by adding a new source s below C_1. Then, s is attached it to any vertex of C_1 by an edge, which can be drawn upward. Otherwise, we choose any source s of S_0 which is visible from the lower border of the cylinder. Then, we eliminate all other sources of S_0 as described in [21], which is the technique used in the proof of Lemma 4.3. The graph consisting of S_0, the vertices of C_1, and the edges between vertices from S_0 and C_1 is **SUP**. Accordingly, we proceed for S_{k+1} above C_k and may add a new sink t if S_{k+1} is empty. For each subgraph S_i between two cycles C_i and C_{i+1} we eliminate all sources and sinks by shooting rays and adding an upward drawn edge, as explained in the proof of Lemma 4.3. Without the edges from the cycles S_i is acyclic, and in the compound graph it becomes a transit. Let H be the resulting graph, which is **wSUP**. H is a pathway since its compound graph is a path from



(a) A **wSUP**-embedded graph.

(b) Component and compound graph of the graph from Fig. 13(a).



(c) The source s and the dashed edges have been introduced to obtain a **wSUP** graph with exactly one source and one sink.

(d) The component and compound graph, where the latter is a path from s to t and, hence, the whole graph is a pathway.

(e) The dual of the graph in Fig. 13(c) with its compound graph. The dual is a **RUP**-embedded closed graph, where the transits consist of edges only.

Figure 13: A **wSUP** graph and its augmentation to a **wSUP** pathway. Fig. 13(e) shows the dual of the augmented graph.

s to t and there are transits τ_i from s to C_1, from C_k to t and from C_i to C_{i+1} for each subgraph s_i with $1 \leq i < k$, and the compounds are disjoint simple cycles.

"$\Leftarrow$": Let $H = (s, \tau_1, \gamma_1, \ldots, \gamma_{k-1}, \tau_k, t)$ for $k \geq 0$ be a planar pathway with simple cycles γ_i as compounds. Then its component graph $\mathbb{H}$ is a dipole and thus **SUP**. Let Γ_H be a **SUP** drawing of $\mathbb{H}$, which is expanded to a **wSUP** drawing of G, such that all vertices of a cycle C_i have the same y-coordinate in Γ_H, the edges between two vertices of C_i are drawn horizontally, and the edges that attach vertices of C_i to other vertices not of C_i are drawn upward planar. Then H is drawn **wSUP**, and G is **wSUP**, since **wSUP** is closed unter taking subgraphs. □

Corollary 5.2. *A **wSUP** graph has a supergraph whose dual is **RUP**.*

The converse is not true since the disjoint cycles specialize the transits in the dual G^* to a "bundle" of parallel edges. This also implies that all faces belong to compounds of G^*.

Theorem 5.2. *A graph is **wSUP** if and only if it is the subgraph of a graph whose dual is a **RUP** graph with no trivial component.*

Proof. Augment a **wSUP** graph G to a pathway H as shown in Theorem 5.1, which is a **RUP** graph by Theorem 4.1. If H is acyclic, then the dual H^* is strongly connected and, hence, H^* has no trivial components. Otherwise, let γ be a compound of H. γ is a simple cycle and, hence, its dual is a dipole with a single source and a single sink, and one or more edges in between. Corollary 4.2 and the proof of Lemma 4.2 assert that the dual of γ is a transit of H^*. Hence, each face in H^* belongs to a compound since the transits consist of edges only and H^* contains no trivial components.

Conversely, suppose that G is a subgraph of an embedded graph H, whose dual H^* is **RUP**-embedded and has no trivial components. By Theorem 4.1, H is a pathway. Further, each face in H^* belongs to a compound and, thereby, none of the transits of H^* contains a face besides its source and sink. Hence, the primal of each transit of H^* is a simple cycle in H. Since all cycles in H are simple, G is **wSUP** by Theorem 5.1. □

6. Conclusion

In this paper, we have characterized upward planar graphs in the plane and on the standing and rolling cylinders in terms of their duals. There is a duality between strong connectivity and acyclicity, compounds and transits, and sink and source and closed graphs. In principal, the rolling and the standing cylinder switch roles when going from the primal to the dual. **UP** is closed under taking duals, if the direction of the dual st-edge is reversed. Otherwise, its dual has cycles and is **RUP**.

An asterisk is commonly used to denote a dual, such that G^* is the dual of a (planar, embedded) graph G. We would like to extend this notation to classes of graphs such that $X^* = \{G^* \mid G \in X\}$ for a class of graphs X. Then $X = X^*$ holds for the (undirected) planar graphs X. However, for upward planar graphs this notation only holds under strong restrictions. By $\mathbf{UP} = \mathbf{UP}^*$ we would like to express that the dual of an upward planar graph is upward planar. However, this holds only for st-graphs and with the convention that the dual of the st-edge is reversed. A shortcut for Theorem 4.2 is $\mathbf{RUP}^* = \mathbf{SUP}$ and $\mathbf{SUP}^* = \mathbf{RUP}$, which holds only for strongly connected **RUP** graphs and **SUP** dipoles.

Deciding whether a digraph is upward planar is $\mathcal{NP}$-hard in the plane and on standing and rolling cylinders [10, 19, 23]. However, if one gets hold of an embedding, upward planarity becomes linear-time solvable. In particular, for strongly connected graphs, we have devised a linear-time algorithm in [4] that computes a **RUP** embedding or rejects if the graph is not **RUP**. This algorithm was extended to work for closed graphs in [1].

Acknowledgement

We would like to thank the anonymous referees for their useful comments.

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