The Density of Classes of 1-Planar Graphs*

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The *density* of a graph $G = (V, E)$ is the number of edges $|E|$ as a function of the number of vertices $n = |V|$. It is an important graph parameter, and is often used to exclude a graph from a particular class. We survey the density of relevant subclasses of 1-planar graphs and establish some new and improved bounds. A graph is *1-planar* if it can be drawn in the plane such that each edge is crossed at most once.

We consider simple and connected graphs. A graph $G \in \mathcal{G}$ is *maximal* for a particular class of graphs $\mathcal{G}$ if the addition of any edge e implies $G + e \notin \mathcal{G}$. Let $M(\mathcal{G}, n)$ and $m(\mathcal{G}, n)$ denote the *maximum* and *minimum numbers* of edges of a maximal n-vertex graph in $\mathcal{G}$. Graphs $G \in \mathcal{G}$ with density $M(\mathcal{G}, |G|)$ ($m(\mathcal{G}, |G|)$) are the *densest* (*sparsest maximal*) graphs of $\mathcal{G}$. Thus $M(\mathcal{G}, n)$ is an upper and $m(\mathcal{G}, n)$ a lower bound. It is well-known that $M(\mathcal{G}, n)$ and $m(\mathcal{G}, n)$ coincide for planar, bipartite planar, and outerplanar graphs with $3n-6$, $2n-4$, and $2n-3$, respectively. For 1-planar graphs the upper and lower bounds diverge. $M(\mathcal{G}, n) = 4n - 8$ was proved first of all by Bodendiek et al. [3] and was rediscovered several times. Surprisingly, there are much sparser maximal 1-planar graphs that are even sparser than maximum planar graphs. In [4] it was proved that $\frac{28}{13}n - \mathcal{O}(1) \leq m(\mathcal{G}, n) \leq \frac{45}{17}n - \mathcal{O}(1)$.

We consider the density of maximal graphs of subclasses of 1-planar graphs, with emphasis on sparse graphs. Our focus is on 3-connected [1], bipartite [7,8], and outer 1-planar [2] graphs. An *outer 1-planar* graph is drawn with all vertices in the outer face. Moreover, we restrict the drawings by fixed *rotation systems*, which specify the cyclic ordering of the edges at each vertex, and then may allow crossings of incident edges, which are generally excluded for 1-planarity.

Theorem 1. *For the classes of graphs $\mathcal{G}$ from Table 1, the stated upper bound on $M(\mathcal{G}, n)$ on the density is tight. The minimum density $m(\mathcal{G}, n)$ ranges between the functions in column "lower example" and "lower bound m".*

3-connected. For 3-connected 1-planar graphs $\mathcal{G}$ the upper bound is obvious. The lower bound $m(\mathcal{G}, n) = \frac{10}{3}n + \frac{20}{3}$ is tight. It improves the example of $3.625n + \mathcal{O}(1)$ from [6] and disproves their conjecture of $3.6n + \mathcal{O}(1)$.

A graph $G \in \mathcal{G}$ consists of non-planar K_4s and a planar remainder, which is triangulated such that two adjacent triangles imply a K_4. The removal of all pairs of crossing edges from G leaves a planar graph with t triangles and q quadrangles and the relation $t \leq q$, which together with Euler's formula yields the

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	upper bound M	lower example	lower bound m
2-connected	$4n-8$ [3]	$\frac{45}{17}n-\frac{84}{17}$ [4]	$\frac{28}{13}n-\frac{10}{3}$ [4]
3-connected	$4n-8$ [3]	$\frac{10}{3}n-\frac{20}{3}$	$\frac{10}{3}n-\frac{20}{3}$
straight-line	$4n-9$ [5]	$\frac{8}{3}n-\frac{11}{3}$	$\frac{28}{13}n-\frac{10}{3}$ [4]
fixed rotation, 2-connected	$4n-8$ [3]	$\frac{7}{3}n-3$ [4]	$\frac{21}{10}n-\frac{10}{3}$ [4]
fixed rotation, intersect incident	$4n-8$ [3]	$\frac{3}{2}n+1$	$\frac{5}{4}n$
bipartite	$3n-8$ [7]	$n-1$	$n-1$
bipartite, 2-connected	$3n-8$ [7]	$2n-4$	n
outer 1-planar	$\frac{5}{2}n-4$	$\frac{11}{5}n-\frac{18}{5}$	$\frac{11}{5}n-\frac{18}{5}$

Table 1. Upper and lower bounds on the number of edges in maximal graphs

bound for $m(\mathcal{G}, n)$. The bound is achieved by a recursive construction of planar K_4s surrounded by non-planar K_4s surrounded by planar K_4s.

Outer 1-planar. A maximal outer 1-planar graph G is composed of planar K_3s and non-planar K_4s [2], such that two K_3s are not adjacent. Removing the pairs of crossing edges from the K_4s results in an outerplanar graph whose dual is a tree with vertices of degree 3 and 4. Each vertex of degree 3 adds one vertex and two edges, and each vertex of degree 4 adds two vertices and five edges to the density of G. Maximizing the degree-4 vertices yields $M(\mathcal{G}, n) = \frac{5}{2}n - 4$ and minimizing yields $m(\mathcal{G}, n) = \frac{11}{5}n - \frac{18}{5}$. Both bounds are tight.

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