The Challenges of Non-linear Parameters and Variables in Automatic Loop Parallelisation

Armin Größlinger
December 2, 2009

Rigorosum

Fakultät für Informatik und Mathematik
Universität Passau

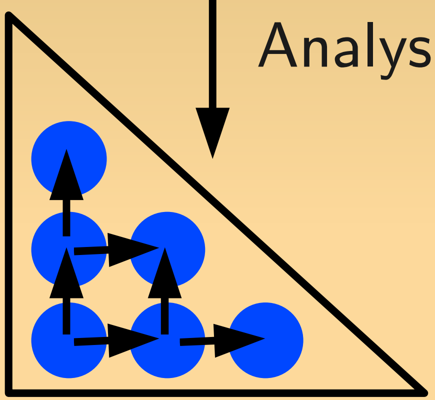
Automatic Loop Parallelisation

```
for (i=1; i<=n; i++)
   for (j=1; j<=n-i; j++)
      A[i][j]=A[i-1][j]+A[i][j-1];
```

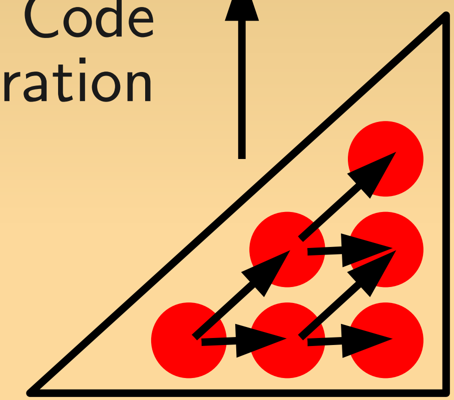
```
for (t=1; t<=n; t++)
   parfor (p=1; p<=t; p++)
      A[t-p+1][p] = ...;
```

Analysis

Code generation



Transformation(s)



$1 \leq i \leq n$

$1 \leq j \leq n - i$

Loop bounds and array indices are **linear** (affine) **expressions.**

→ **Polyhedron model**

$1 \leq t \leq n$

$1 \leq p \leq t$

Dependences:

$(i, j) \to (i+1, j)$

$(i, j) \to (i, j+1)$

$(t, p) \to (t+1, p)$

$(t, p) \to (t+1, p+1)$

Non-linearity?

The polyhedron model can handle some codes in, e.g.,

- Simulation, image processing, linear algebra.

Today, parallelism is everywhere:

- Multi-core CPUs, many-core CPUs, graphics card computing (GPGPU)
- Automatic parallelisation helps not to burden software developers with the parallelism.
- Non-linearities make the polyhedron model more widely applicable:
  - Handle more programs,
  - Target more diverse hardware.

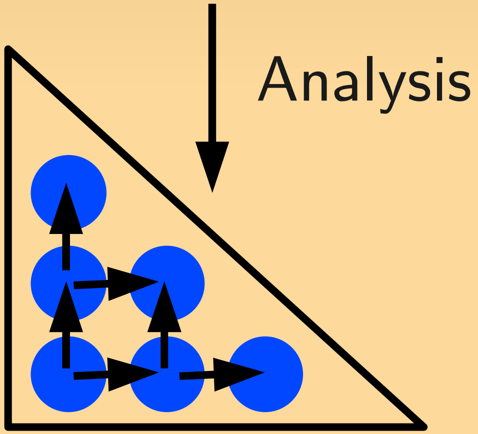
Non-linearity

- Linear: A[2*i + 3*j − 4*m + 5*n + 7]
  expressions linear in the variables *and* the parameters.
- Non-linearity:
  - A[n*i + m*m*j + n*m]

    Expressions still linear in the variables
    ("non-linear parameters").
  - A[i*j + m*j*j]

    Arbitrary polynomials in the variables and parameters.

```
for (i=1; i<=n; i++)
   for (j=1; j<=n-i; j++)
      ...
```



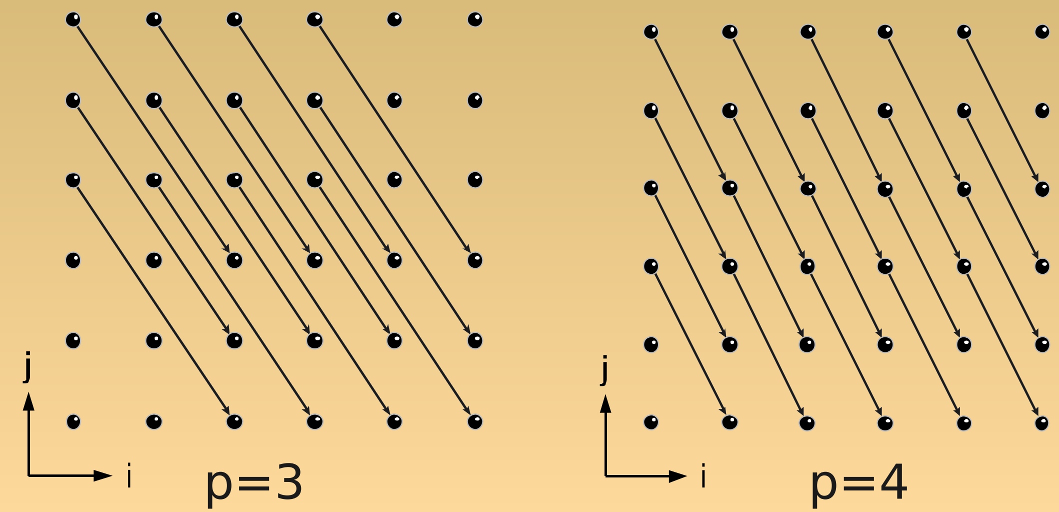
Part 1: Non-linearity in Dependence Analysis

Dependence Analysis Example

```
for (i=0; i<=m; i++)
    for (j=0; j<=m; j++)
        ... A[p*i+2*j] ...
```

"When is A[x] accessed again?"

Which iterations (i,j) access the same array element?



Result of our automatic analysis:

$$(i,j) \to (i+1, j-\tfrac{p}{2}) \quad \text{if} \begin{cases} p \equiv_2 0, m \geq 1, -2m \leq p \leq 2m, 0 \leq i \leq m-1, \\ \max(0,\tfrac{p}{2}) \leq j \leq \min(m, m+\tfrac{p}{2}) \end{cases}$$

$$(i,j) \to (i+2, j-p) \quad \text{if} \begin{cases} p \equiv_2 1, m \geq 2, -m \leq p \leq m, 0 \leq i \leq m-2, \\ \max(0,p) \leq j \leq \min(m, m+p) \end{cases}$$

(Trying to use weak quantifier elimination in the integers to compute the dependences yields an output with > 20,000 lines.)

A Non-linear Parameter Example

```
for (i=0; i<=m; i++) {
    for (j=0; j<=n; j++) {
        ... A[4*i+2*j] ...
    }
    ... A[p*i+1] ...
}
```

$$4 \cdot i + 2 \cdot j = p \cdot i' + 1$$

$$(i\ j\ i') \begin{pmatrix} 4 \\ 2 \\ -p \end{pmatrix} = 1$$

Solutions for i, j, $i' \in \mathbb{Z}$ in dependence of $p \in \mathbb{Z}$?

For $p \equiv_2 0$: no solution.

For $p \equiv_2 1$:

$$i = t_1$$

$$j = (-2p - 2) \cdot t_1 - p \cdot t_2 - \frac{p+1}{2}$$

$$i' = -4t_1 - 2t_2 + 1$$

for $t_1, t_2 \in \mathbb{Z}$.

Linear Diophantine Equation Systems

To solve a system of linear Diophantine equations

$$x \cdot A = b \qquad \text{with } A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^n$$

for $x \in \mathbb{Z}^m$, all we need is an algorithm to compute GCDs.

(More precisely, for $c, d \in \mathbb{Z}$, we must be able to compute $g, u, v \in \mathbb{Z}$ such that: $\gcd_{\mathbb{Z}}(c, d) = g = u \cdot c + v \cdot d$.)

Result: We can perform a similar procedure when A and b depend on $p \in \mathbb{Z}$, i.e., we want to solve

$$x \cdot A(p) = b(p)$$

for x in dependence of p.

Armin Größlinger and Stefan Schuster. On Computing Solutions of Linear Diophantine Equations with One Non-linear Parameter. In *Proc. of SYNASC 2008*, pages 69–76. IEEE Comp. Soc., 2009.

Generalisation

How do we generalise the classical procedure to solve

$$(i\ j\ i') \begin{pmatrix} 4 \\ 2 \\ -p \end{pmatrix} = 1 \qquad ?$$

What is the GCD of 2 and p? $\gcd_{\mathbb{Z}}(2, p) = \begin{cases} 2 & \text{if } p \equiv_2 0 \\ 1 & \text{if } p \equiv_2 1 \end{cases}$

Modelling p by the unknown X of $\mathbb{Z}[X]$ does not work:
$\gcd_{\mathbb{Z}[X]}(X, 2) = 1 \qquad ↯$

$$\gcd_{\mathbb{Z}[X]}(f, g)(p) \neq \gcd_{\mathbb{Z}}\big(f(p), g(p)\big)$$ (in general)

"polynomial GCD" "pointwise GCD"

Is there a polynomial ring $\supseteq \mathbb{Z}[X]$ in which polynomial and pointwise GCD coincide?

Entire Quasi-polynomials

Definition. A function $c : \mathbb{Z} \to \mathbb{Q}$ with period $l \geq 1$, i.e., $\forall p \in \mathbb{Z} : c(p) = c(p + l)$ is called a *periodic number*.
Notation: $[c(0), \ldots, c(l-1)]$, e.g., $[1, 2, 3]$.

Definition. $f = \sum_{i=0}^{u} c_i X^i$ with periodic numbers c_i as coefficients is called a *quasi-polynomial*.
Evaluation: $f(p) := \sum_{i=0}^{u} c_i(p) \cdot p^i$ for $p \in \mathbb{Z}$.

Entire quasi-polynomials: $EQP = \{f \mid \forall p \in \mathbb{Z} : f(p) \in \mathbb{Z}\}$

Example:

$$f = [\tfrac{3}{2}, \tfrac{1}{2}]X + [1, \tfrac{1}{2}] \in EQP$$

because $f(1) = \frac{1}{2} \cdot 1 + \frac{1}{2} = 1$, $f(2) = \frac{3}{2} \cdot 2 + 1 = 4$, etc.

Division with Remainder in EQP

- GCDs can be computed using division with remainder.
- We can define a kind of division with remainder in EQP, e.g.:

$$X^2 = \left(\tfrac{1}{2}X - [0, \tfrac{1}{2}]\right) \cdot 2X + [0,1]X$$

- Only complication: zero-divisors.
  No divisions in components that are zero.

GCDs in EQP

This division in EQP allows to construct **finite** remainder sequences:

$$
\begin{aligned}
f_0 &= q_0 \cdot f_1 + f_2 \\
f_1 &= q_1 \cdot f_2 + f_3 \\
&\vdots \\
f_{n-1} &= q_{n-1} \cdot f_n \\
&\Downarrow \\
f_n &= \gcd_{EQP}(f_0, f_1)
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
f_0(p) &= q_0(p) \cdot f_1(p) + f_2(p) \\
f_1(p) &= q_1(p) \cdot f_2(p) + f_3(p) \\
&\vdots \\
f_{n-1}(p) &= q_{n-1}(p) \cdot f_n(p) \\
&\Downarrow \\
f_n(p) &= \gcd_{\mathbb{Z}}\big(f_0(p), f_1(p)\big)
\end{aligned}
$$

$$
\boxed{\gcd_{EQP}(f_0, f_1)(p) = \gcd_{\mathbb{Z}}\big(f_0(p), f_1(p)\big)}
$$

Weak and Pointwise Echelon Form

$S_1 = \begin{pmatrix} [1,0]X & 1 \\ 0 & 1 \end{pmatrix}$ is in echelon form, because $[1,0]X \neq 0$ and $1 \neq 0$.

But $S_1(p)$ is *not* echelon for $p = 0$, $p \equiv_2 1$: $S_1(p) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

Serious problem: periodically vanishing pivots

Solution:
Additional row operations in the vanishing components.

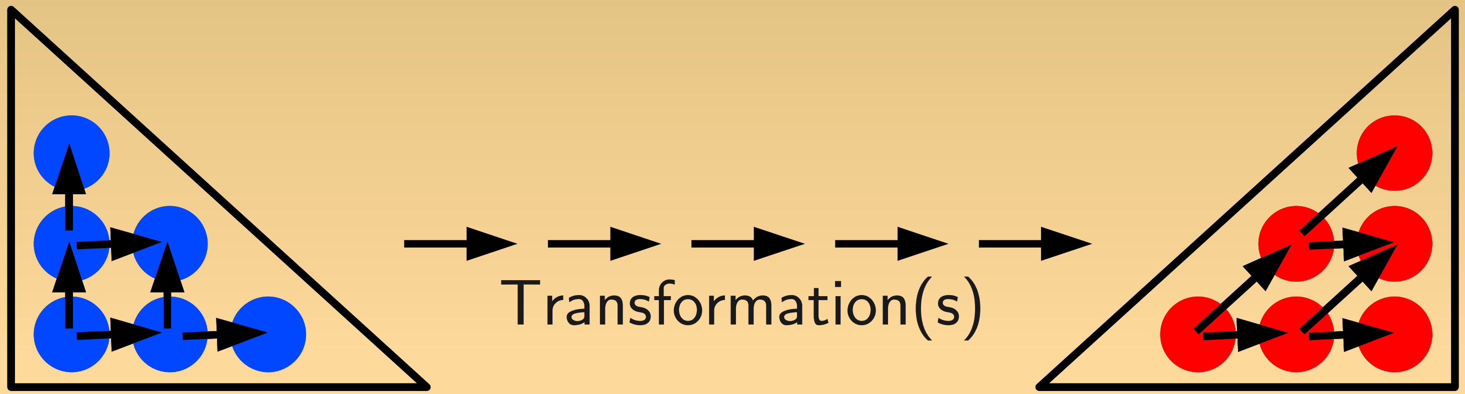
$$S_1 \rightsquigarrow S_2 = \begin{pmatrix} [1,0]X & 1 \\ 0 & [1,0] \end{pmatrix}$$

subtract first row times $[0,1]$ from second row

$S_2(p)$ is echelon for all $p \in \mathbb{Z} - M$, $M = \{0\}$.

Dependence Analysis Summary

- Entire quasi-polynomials allow to compute pointwise solutions of a system of linear Diophantine equations with one non-linear parameter.
- This also generalises Banerjee's data dependence to one non-linear parameter.
- Previously, only syntactic treatment of non-linearities (Pugh et al. 1995) or approximations.



Part 2:
Non-linearities in Transformations

Non-linear Transformations

- Transformations may introduce non-linearities for different reasons, e.g.:
  - Explicit non-linear schedules which are better than the best linear schedules (Achtziger et al. 2000),
  - Non-linear parameter models a compile-time unknown (e.g. number of processors for tiling for a variable number of processors).

Quantifier Elimination vs Algorithm + QE

- Some transformations (e.g., computing a schedule) can be expressed as quantifier elimination (QE) or QE with answer problems.
- Unfortunately, QE is too slow even for small examples.
- Alternative: Enhance a classical algorithm with the help of QE to handle non-linear parameters. Successful for, e.g.,
  - Fourier-Motzkin elimination,
  - Simplex,
  - Chernikova's algorithm.

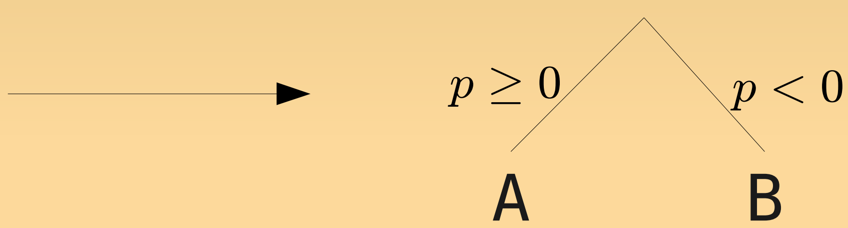
Armin Größlinger, Martin Griebl, and Christian Lengauer. Quantifier Elimination in Automatic Loop Parallelization. *Journal of Symbolic Computation*, 41(11):1206–1221, Nov. 2006.

Classical Algorithm + QE

- Classical algorithms (like Simplex) make case distinctions on the signs of values in a coefficient matrix:

$$\begin{pmatrix} 1 & 2 & -4 & 0 \end{pmatrix} \qquad \begin{pmatrix} p & p^2 - q & -p & 0 \end{pmatrix}$$

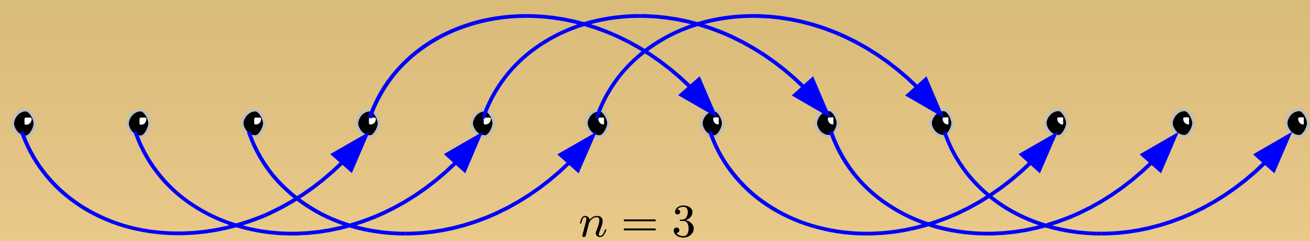
```
if c >= 0 then
   A
else
   B
```



- With non-linear parameters, values are symbolic expressions in the parameters.
  → Case distinctions in the result.
- QE is used to prune paths with inconsistent conditions.
- Correctness by construction.
- Termination has to be proved.

Scheduling Example

Dependence:
$i \to i + n$



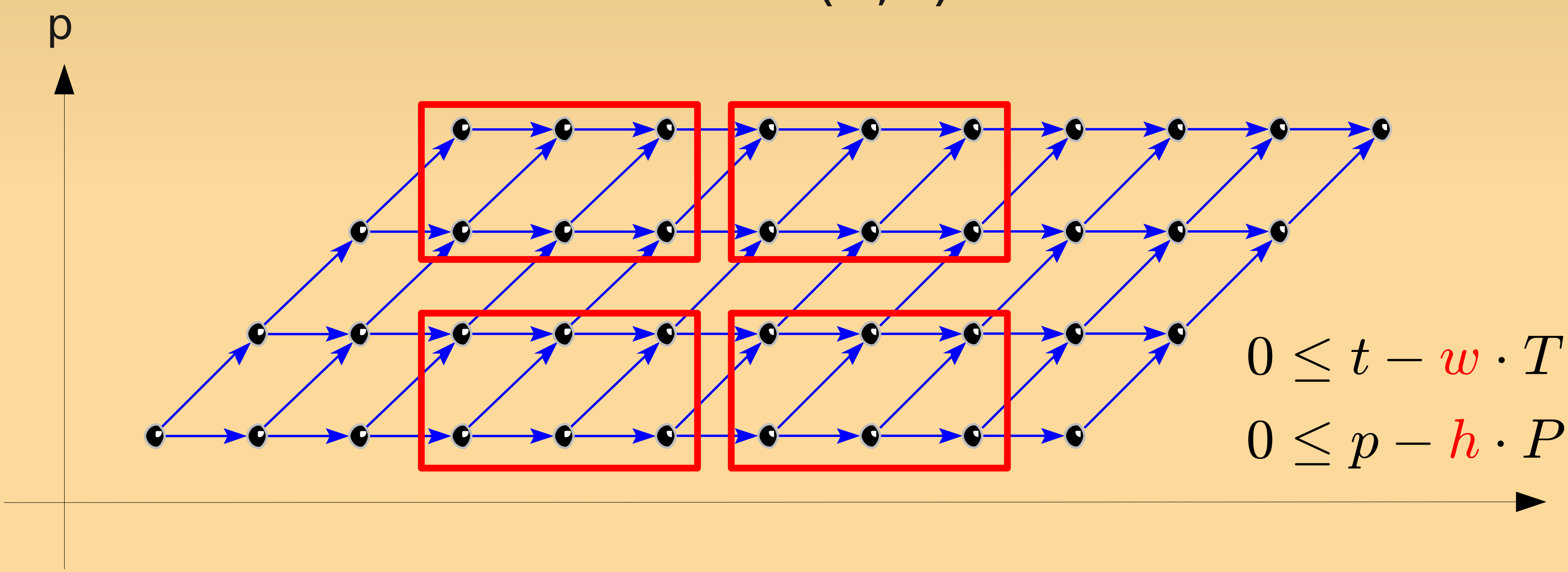
Desired schedule: $\theta(i) = \lfloor \frac{i}{n} \rfloor$

Observations:

- Both QE with answer and Simplex+QE compute the desired schedule in a short time.
  (about 2 seconds on Core2Duo 2.4 GHz)
- QE with answer fails (is too slow or uses too much memory) for more complex examples
  (2-dimensional iteration domain, 2 dependences).

Tiling

- The parallelism often has to be coarsened by grouping operations into bigger chunks.
- Example: tiles with width w and height h;
Coordinates of the tiles: (T,P)



$$0 \leq t - w \cdot T \leq w - 1$$
$$0 \leq p - h \cdot P \leq h - 1$$

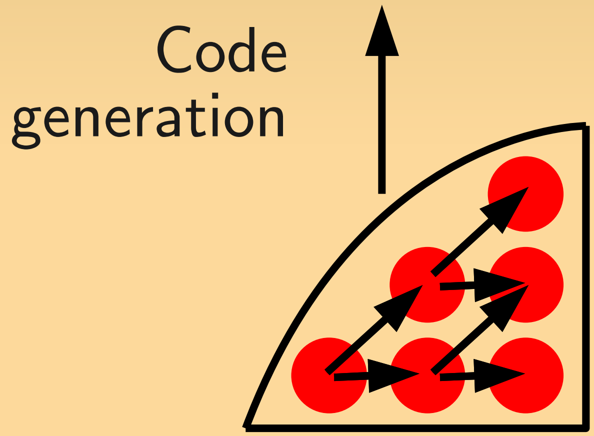
Armin Größlinger. Some Experiments on Tiling Loop Programs for Shared-Memory Multicore Architectures.
Dagstuhl seminar number 07361 proceedings, 2008.

Transformations Summary

- Non-linear transformations are becoming more desirable as we try to apply the polyhedron model to a wider range of programs or hardware.
- Even ”harmless” transformations may cause non-linearities to appear.

Part 3: Code Generation for Non-linearly Bounded Iteration Domains

```
for (t=1; t<=n; t++)
   parfor (p=1; p<=n-(n-t)^2; p++)
      ...;
```

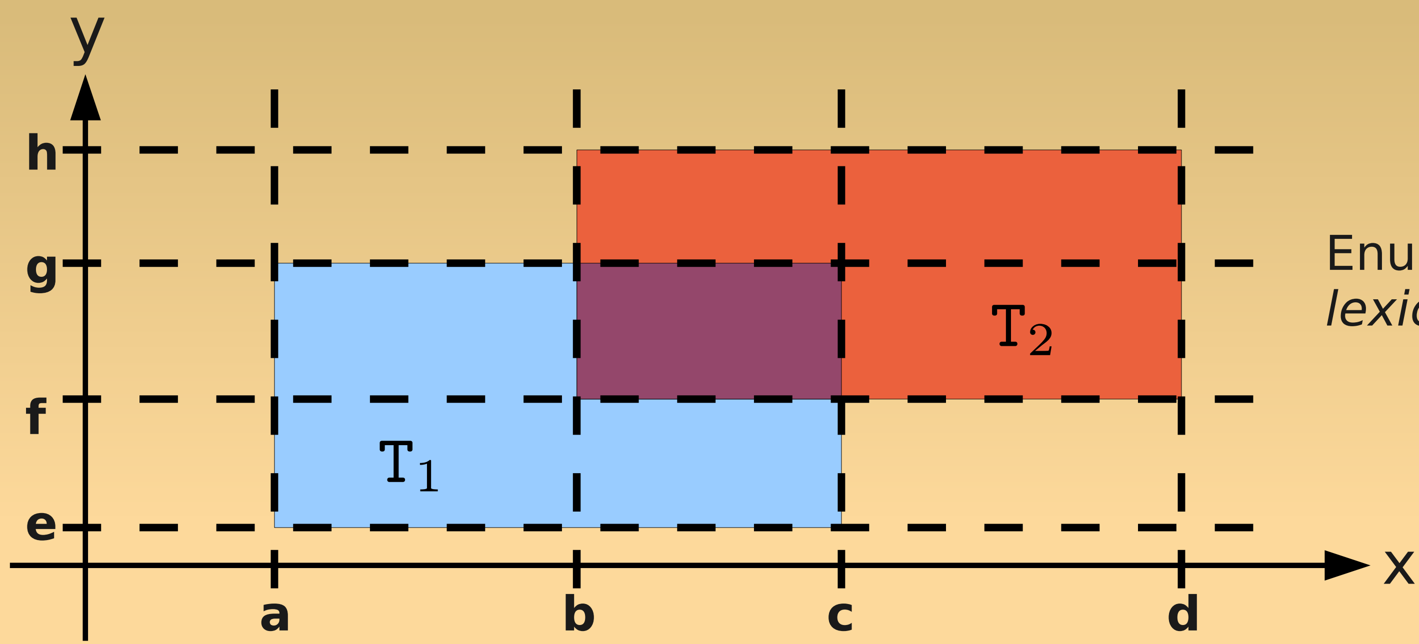


Non-linear Code Generation?

- Why non-linear code generation?
  - Non-linear parameters and variables are introduced by transformations (cf. Part 2).
- A single non-linearity makes it impossible to use current code generation techniques (e.g., Bastoul 2004).

Armin Größlinger. Scanning Index Sets with Polynomial Bounds Using Cylindrical Algebraic Decomposition. Technical Report MIP-0803, Fakultät für Informatik und Mathematik, Universität Passau, 2008.

The Essence of Code Generation



Enumerate iterations in *lexicographic* order.

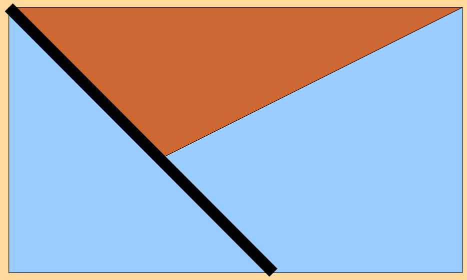
```
for (x=a; x≤d; x++) {
  for (y=e; y≤h; y++) {
    if (a≤x≤c ∧ e≤y≤g) T1;
    if (b≤x≤d ∧ f≤y≤h) T2;
  }
}
```

For efficiency:
No case distinctions inside the loops!

```
for (x=a; x≤b-1; x++)
  for (y=e; y≤g; y++) T1;
for (x=b; x≤c; x++) {
  for (y=e; y≤f-1; y++) T1;
  for (y=f; y≤g; y++) { T1; T2; }
  for (y=g+1; y≤h; y++) T2;
}
for (x=c+1; x≤d; x++)
  for (y=f; y≤h; y++) T2;
```

Polyhedral Code Generation

- Compute partitionings of the iteration domains and their projections onto outer dimensions by
  - intersections and differences of polyhedra,
  - projections of polyhedra.
- Invariant: intersections, differences and projections yield finite unions of polyhedra.
  → finitely many convex sets



- Partitions (polyhedra) can be ordered in each dimension. The choice of the partitioning only affects the efficiency of the generated code.

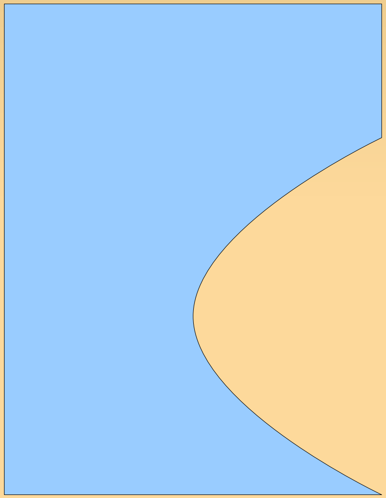
Loops for Polyhedra with Non-linear Parameters

- Using QE we can generalise polyhedral code generation to non-linear parameters:
  - Fourier-Motzkin (or Chernikova) used to compute projections.
  - QE used to compute disjoint unions of polyhedra and ordering of polyhedra.
- The prototype implementation can generate code for all examples in CLooG's test suite.

Loops for Semi-algebraic Iteration Domains

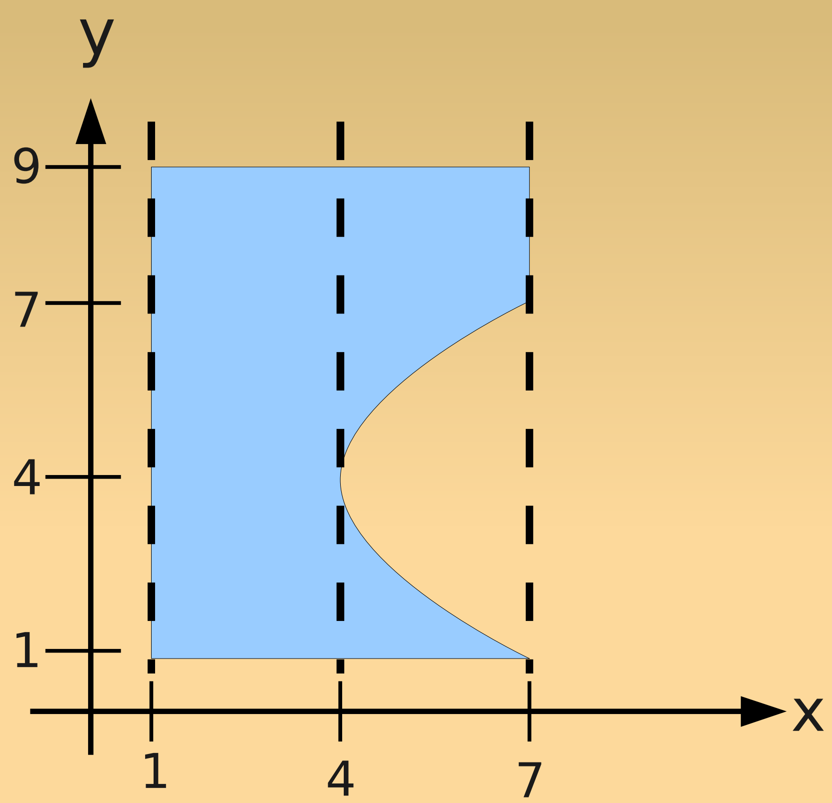
- Semi-algebraic set =
  defined by polynomial (in-)equalities
- Can be non-convex:

$$1 \leq x \leq 7$$
$$1 \leq y \leq 9$$
$$0 \leq (y-4)^2 + 12 - 3x$$



- Convexity is not necessary for code generation.
- The analogy to dimension-wise ordered convex sets is ***cylindrical*** (algebraic) ***decomposition***.

A Semi-algebraic Example




```
for (x=1; x≤4; x++)
   for (y=1; y≤9; y++)
      T(x,y);
for (x=4+1; x≤7; x++) {
   for (y=1; y≤⌊4−√(3x−12)⌋; y++)
      T(x,y);
   for (y=⌈4+√(3x−12)⌉; y≤9; y++)
      T(x,y);
}
```

$$1 \leq x \leq 7$$

$$1 \leq y \leq 9$$

$$0 \leq (y-4)^2 + 12 - 3x$$

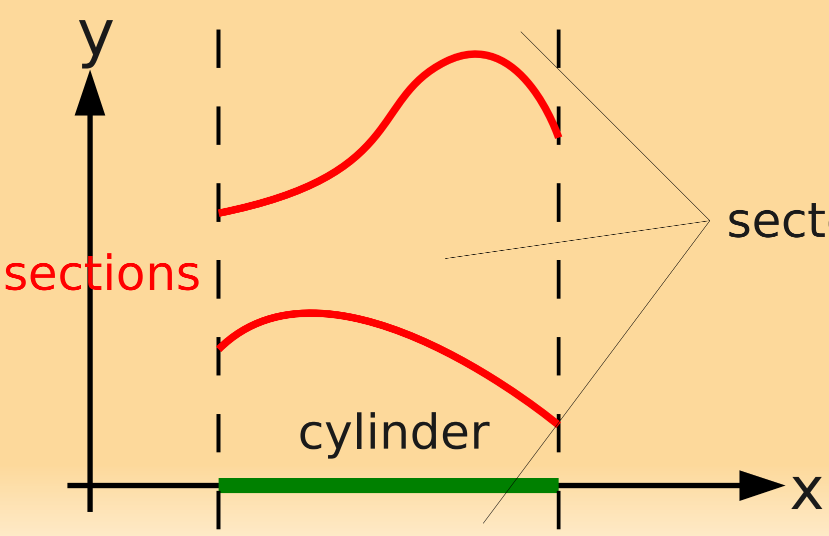
Cylindrical Decomposition

Let $R \subseteq \mathbb{R}^n$ connected, $R \neq \varnothing$.

Then $R \times \mathbb{R}$ is called a *cylinder* over R.

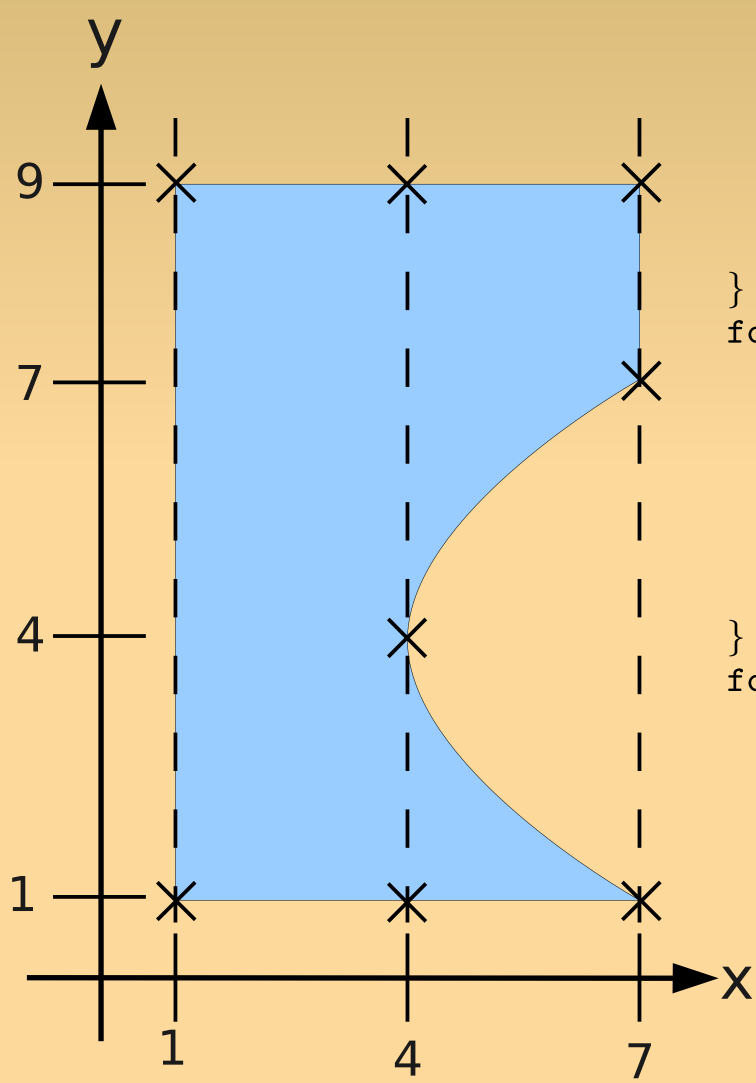
Let $f_1, \ldots, f_r : R \to \mathbb{R}$ continuous
and $\forall x \in R : f_1(x) < f_2(x) < \cdots < f_r(x)$.

Then $(f_1, \ldots, f_r)$ defines a *stack* over R.
The graphs of the f_i are called *sections*, and
the regions between the graphs are called a *sectors*.



Cylindrical algebraic decomposition: f_i are roots of (multi-variate) polynomials.

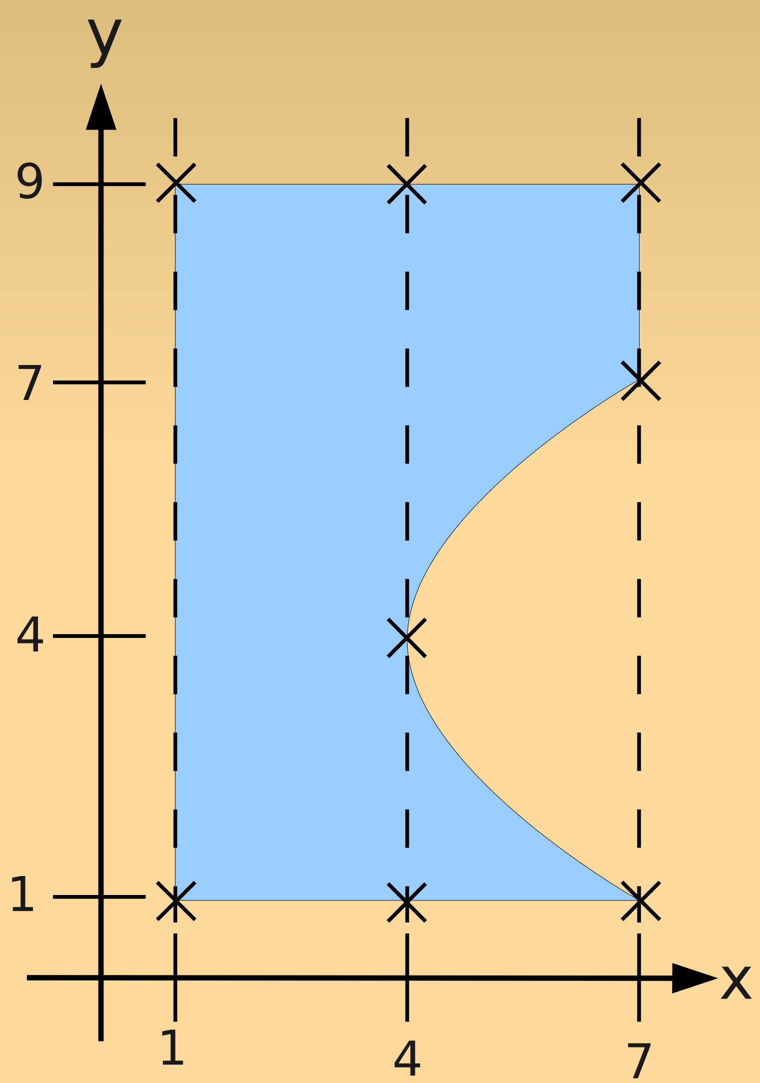
Code for the Example




```
for (x=1; x≤1; x++) {
  for (y=1; y≤1; y++)
    T(x,y);
  for (y=1+1; y≤9-1; y++)
    T(x,y);
  for (y=9; y≤9; y++)
    T(x,y);
}
for (x=1+1; x≤4-1; x++) {
  for (y=1; y≤1; y++)
    T(x,y);
  for (y=1+1; y≤9-1; y++)
    T(x,y);
  for (y=9; y≤9; y++)
    T(x,y);
}
for (x=4; x≤4; x++) {
  for (y=1; y≤1; y++)
    T(x,y);
  for (y=1+1; y≤4-1; y++)
    T(x,y);
  for (y=4; y≤4; y++)
    T(x,y);
  for (y=4+1; y≤9-1; y++)
    T(x,y);
  for (y=9; y≤9; y++)
    T(x,y);
}
```

```
for (x=4+1; x≤7-1; x++) {
  for (y=1; y≤1; y++)
    T(x,y);
  for (y=1+1; y≤⌈4−√(3x−12)⌉ − 1; y++)
    T(x,y);
  for (y=⌈4−√(3x−12)⌉; y≤⌊4−√(3x−12)⌋; y++)
    T(x,y);
  for (y=⌈4+√(3x−12)⌉; y≤⌊4+√(3x−12)⌋; y++)
    T(x,y);
  for (y=⌊4+√(3x−12)⌋ + 1; y≤9-1; y++)
    T(x,y);
  for (y=9; y≤9; y++)
    T(x,y);
}
for (x=7; x≤7; x++) {
  for (y=1; y≤1; y++)
    T(x,y);
  for (y=⌈4+√(3x−12)⌉; y≤⌊4+√(3x−12)⌋; y++)
    T(x,y);
  for (y=⌊4+√(3x−12)⌋+1; y≤9-1; y++)
    T(x,y);
  for (y=9; y≤9; y++)
    T(x,y);
}
```

Simplified Code




```
for (x=1; x≤4; x++) {
  for (y=1; y≤9; y++)
    T(x,y);
}
for (x=4+1; x≤7; x++) {
  for (y=1; y≤⌊4−√(3x − 12)⌋; y++)
    T(x,y);
  for (y=⌈4+√(3x − 12)⌉; y≤9; y++)
    T(x,y);
}
```

Generated code can be simplified automatically.

Code Generation Summary

- QE allows to generalise polyhedral code generation to non-linear parameters.
- Cylindrical decomposition enables to generate code for arbitrary semi-algebraic iteration domains.
- Prototypical implementations available:
  - Using FM/Chernikova+QE: NLGen
    (to be released soon).
    Can generate code for all of CLooG's test cases.
  - Using CAD: CADGen version 0.1, available at
    https://www.infosun.fim.uni-passau.de/trac/LooPo/wiki/CADGen
    Can generate code for a few of CLooG's test cases.
- Open question: relation of code generation to formula simplification (e.g., GEOFORM formulas)?

Conclusions

- The applicability of automatic loop parallelisation is restricted by many cases that are ”slightly” outside the polyhedron model.
- In all three phases of the parallelisation process non-linearities can be handled.
- Dependence analysis is most challenging.
- Code generation is solved in theory.
- Quantifier elimination with answer is often too general and, therefore, too slow.
- Combining polyhedral methods (for polyhedral sub-problems) with the more general ones may improve the efficiency.