1-Gap Planarity of Complete Bipartite Graphs

Christian Bachmaier, Ignaz Rutter, and Peter Stumpf

Faculty of Computer Science and Mathematics, University of Passau
{bachmaier,rutter,stumpf}@fim.uni-passau.de

Abstract. A graph is 1-gap planar if it admits a drawing such that each crossing can be assigned to one of the two involved edges in such a way that each edge is assigned at most one crossing. We show that $K_{3,14}$, $K_{4,10}$ and $K_{6,6}$ are not 1-gap planar.

1 Introduction

A graph is 1-gap planar if it admits a drawing such that each crossing can be assigned to one of the two involved edges in such a way that each edge is assigned at most one crossing. The motivation comes from edge casings, where one creates a small gap in one of the edges involved in each crossing to increase the readability. In a 1-gap planar drawing each edge receives at most one such gap. This notion was introduced in GD'17 by Bae et al. [1]. Among others they showed that a 1-gap planar graph on n vertices has at most $5n - 10$ edges and this is tight. They further show that that the complete graph K_n is 1-gap planar if and only if $n \leq 8$. An important observation of Bae et al. is that every 1-gap planar graph G satisfies $\mathrm{cr}(G) \leq |E|$ (since each crossing is assigned to one of the edges). For complete bipartite graphs, they gave 1-gap planar drawings for $K_{3,12}$, $K_{4,8}$ and $K_{5,6}$, whereas they exclude $K_{3,15}$, $K_{4,11}$ and $K_{6,7}$ by observing that their crossing number is strictly greater than their edge number. They leave the remaining complete bipartite graphs as an open problem. We show the following theorem.

Theorem 1. *The graphs $K_{3,14}$, $K_{4,10}$ and $K_{6,6}$ are not 1-gap planar.*

This shrinks the open cases to $K_{3,13}$ and $K_{4,9}$. We note that for all the graphs we exclude, the crossing number equals the edge number [2]. Thus, we know that in a 1-gap planar drawing of such a graph each edge has at least one crossing.

2 Proof Strategy

Our proof strategy is an extension of the one of Bae et al., who encountered a similar situation when treating the case of K_9, which has 36 edges and whose crossing number is 36. For convenience, we briefly sketch their argument. Assume for the sake of contradiction that Γ is a 1-gap planar drawing of K_9, and consider the planarization $\Gamma^\star$ of this drawing, where all crossings are replaced by dummy vertices. Observe that Γ has precisely $\mathrm{cr}(K_9) = |E(K_9)| = 36$ crossings [1]. If two vertices of K_9 share a face in $\Gamma^\star$, we can reroute the edge between them without crossings in this face, thus obtaining a drawing with fewer crossings, which is not possible. Thus, for any two original vertices their incident faces of $\Gamma^\star$ are disjoint. This gives a lower bound of 72 faces. On the other hand, from Euler's formula it follows that $\Gamma^\star$ has only 65 faces; a contradiction.

In contrast, for complete bipartite graphs, vertices may share a face of the planarization if they are independent. Let $G = (R \dot\cup B, E)$ be a complete bipartite graph with $\mathrm{cr}(G) = |E| = |R| \cdot |B|$. The vertices in R and B are *red* and *blue*, respectively. As before, we consider a hypothetical 1-gap planar drawing Γ of G, for which we know that it has $\mathrm{cr}(\Gamma) = \mathrm{cr}(G) = |E|$ crossings, and we denote the planarization by $\Gamma^\star$. Let F denote the set of faces of $\Gamma^\star$ and let $F_R, F_B \subseteq F$ be the faces that are incident to a red and a blue vertex, respectively. If $F_R \cap F_B \neq \emptyset$, then there is a face in F that is incident to both a red and a blue vertex. We can route the edge between them without crossings and thus reach a contradiction as in the case of K_9. By assumption, $\Gamma^\star$ has $|R|+|B|+|E|$ vertices and $|R|\cdot|B|+2\cdot|E|$ edges, and hence $|F| = 2 \cdot |R| \cdot |B| - |R| - |B| + 2$ faces.

Consider the auxiliary bipartite graph $G_R = (R \cup F_R, E_R)$ where a face and a vertex are adjacent if and only if they are incident in $\Gamma^\star$. The graph $G_B = (B \cup F_B, E_B)$ is defined analogously. Observe that $|E_R| = |E_B| = |R| \cdot |B|$ since each vertex in R has degree $|B|$ and vice versa. We argue that either G_R and G_B are both trees, or one of them, say G_R, is a cycle decorated with leaves in F_R and the other one, G_B, is a forest with two connected components.

In the former case, we obtain $|E_R| = |R| + |F_R| - 1$, which gives $|F_R| = |R| \cdot |B| - |R| + 1$ and likewise $|F_B| = |R| \cdot |B| - |B| + 1$. Hence $|F_B| + |F_R| = 2 \cdot |B| \cdot |R| - |R| - |B| + 2 = |F|$. In the latter case, the number of faces in F_R decreases by 1, but the number of faces in F_B increases by 1. In all cases we find that $|F_R| + |F_B| = |F|$, i.e., each face of $\Gamma^\star$ is either in F_R or in F_B. A contradiction is reached by showing that there exists at least one *white face* of $\Gamma^\star$ that is not incident to any red or blue vertex.

First it follows from the fact that each edge has a gap that there is a cycle C in $\Gamma^\star$ that only contains dummy vertices. This can be seen as follows. We start in any dummy vertex and follow the edge that does not have its gap there to its own gap. Repeating this step eventually produces the desired cycle C. If all red and blue vertices lie inside (outside) C, then C contains a white face in its exterior (interior). Otherwise it separates a component of G_R from a component of G_B. Further analysis yields a contradiction. The details vary depending on whether G is $K_{3,14}$, $K_{4,10}$ or $K_{6,6}$ as well as on the size and structure of the components that are separated by C.

3 Conclusion

We have shown that $K_{3,14}$, $K_{4,10}$ and $K_{6,6}$ are not 1-gap planar. We leave open the cases of $K_{3,13}$ and $K_{4,9}$. It seems difficult to adapt our proof technique to these cases since their crossing numbers are strictly smaller than their edge number, which results in additional freedom for possible 1-gap planar drawings.

References

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