

# Linear Time Planarity Testing and Embedding of Strongly Connected Cyclic Level Graphs

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**Abstract.** A level graph is a directed acyclic graph with a level assignment for each node. Such graphs play a prominent role in graph drawing. They express strict dependencies and occur in many areas, e. g., in scheduling problems and program inheritance structures.

In this paper we extend level graphs to cyclic level graphs. Such graphs occur as repeating processes in cyclic scheduling, visual data mining, life sciences, and VLSI. We provide a complete study of strongly connected cyclic level graphs. In particular, we present a linear time algorithm for the planarity testing and embedding problem, and we characterize forbidden subgraphs. Our results generalize earlier work on level graphs.

## 1 Introduction

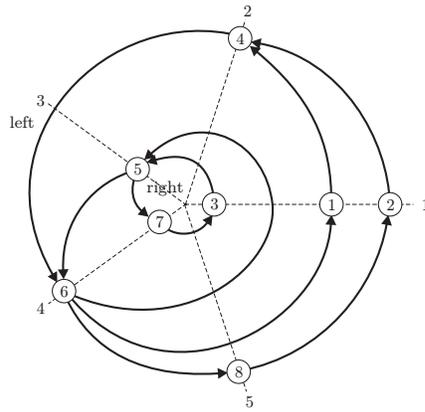
Cyclic level planar graphs receive their motivation from two sources: level planar graphs and recurrent hierarchies. A level graph is a directed acyclic graph with a level assignment for each node. Nodes on the same level are placed at different positions on a horizontal line and edges are drawn downwards from the upper to the lower end node. The challenging problems on level graphs are planarity testing and embedding algorithms in linear time and a characterization in terms of forbidden subgraphs. This parallels the situation for planar graphs, where nowadays there are many  $\mathcal{O}(|V|)$  testing and embedding algorithms [3, 8, 12] and the famous Kuratowski graphs [11]: Each graph is planar if it does not contain a subgraph that is homeomorphic to the complete graph with 5 nodes  $K_5$  or the complete bipartite graph  $K_{3,3}$  with 3 nodes in each set. Level planarity has been studied intensively in recent years [4, 6]. Jünger and Leipert [9] finally established a linear time algorithm for the level planarity testing and embedding problem. Healy et al. [7] gave a complete set of seven level non-planarity patterns for hierarchies. Fowler and Kobourov [5] added two more forbidden graphs for a complete set for arbitrary level graphs. Bachmaier et al. [1] extended level planarity to radial level planarity. There the levels are concentric circles and the edges are directed from inner to outer circles.

Recurrent hierarchies were introduced by Sugiyama et al. [15] more than 25 years ago. A recurrent hierarchy is a level graph with additional edges from the last to the first level. Here two drawings are natural: The first is a 2D drawing, where the levels are rays from a common center, and are sorted counterclockwise by their number, see Fig. 1(a). All nodes of one level are placed at different

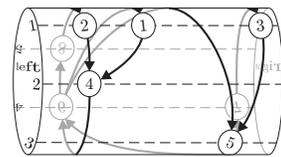
positions on their ray and an edge  $e = (u, v)$  is drawn as a monotone counterclockwise curve from  $u$  to  $v$  wrapping around the center at most once. The second is a 3D drawing on a cylinder, see Fig. 1(b). A planar recurrent hierarchy is shown on the cover of the book by Kaufmann and Wagner [10]. There it is stated that recurrent hierarchies are “unfortunately [...] still not well studied”.

The standard method to visualize directed graphs is the Sugiyama algorithm [15], which eliminates all cycles in the graph. This may be acceptable in many applications. But there are areas in which it is important that the cycles are preserved and represented as cycles, e. g., visual data mining, or chemical reactions in the life sciences [14]. Important applications can further be found in the layout of regular VLSI circuits [13]. Recurrent hierarchies are well suited to visualize such cyclic or regular structures. To enhance the readability of drawings, edge crossings should be avoided and even excluded.

In this paper we improve our earlier result [2] of an  $\mathcal{O}(|V| \log |V|)$  planarity testing and embedding algorithm on strongly connected cyclic level graphs and present an optimal linear time algorithm. Moreover, we characterize forbidden subgraphs for such graphs. This settles the major questions on strongly connected cyclic level graphs.



(a) In the plane



(b) On a cylinder

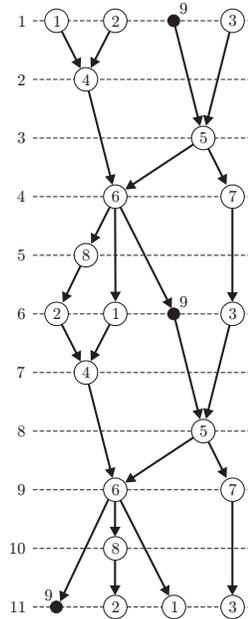


Fig. 2. (2,1)-hierarchy of  $G$

Fig. 1. Drawings of a cyclic 5-level graph  $G$

## 2 Preliminaries

A *cyclic  $k$ -level graph*  $G = (V(G), E(G), \phi_G)$  ( $k \geq 2$ ) is a directed graph without self-loops with a given surjective level assignment of the nodes  $\phi_G: V \rightarrow \{1, 2, \dots, k\}$ . For two nodes  $u_G, v_G \in V(G)$  let  $\text{span}(u_G, v_G) := \phi_G(v_G) - \phi_G(u_G)$  if  $\phi_G(u_G) < \phi_G(v_G)$ , and  $\text{span}(u_G, v_G) := \phi_G(v_G) - \phi_G(u_G) + k$  otherwise. For an edge  $e = (u_G, v_G) \in E(G)$  we define  $\text{span}(e) := \text{span}(u_G, v_G)$ . For a simple path or a simple cycle  $P$  define  $\text{span}(P) := \sum_{e \in E(P)} \text{span}(e)$ . All paths and cycles in this paper are directed if not stated otherwise. A graph is strongly connected if for all  $u_G, v_G \in V(G)$  a path from  $u_G$  to  $v_G$  exists. A cyclic  $k$ -level graph  $G$  is a  *$k$ -level graph* if  $\phi_G(u_G) < \phi_G(v_G)$  for each edge  $(u_G, v_G) \in E(G)$  holds. A drawing is *(cyclic) level plane* if the edges do not cross except on common endpoints. A (cyclic)  $k$ -level graph is *(cyclic) level planar* if such a drawing exists. The *right outer face* is the face of the 2D drawing containing the center and the *left outer face* is the unbounded face (see Fig. 1(a)). A *(cyclic) level planar embedding*  $\mathcal{G}$  of  $G$  consists of two lists  $N_G^-(v_G)$  and  $N_G^+(v_G)$  for each node  $v_G \in V(G)$  which contain the end nodes of ingoing and outgoing edges, respectively. Both lists are ordered from left to right. A *hierarchy* is a level graph  $G$  s. t. each node  $v_G$  with  $\phi_G(v_G) \neq 1$  has an ingoing edge.

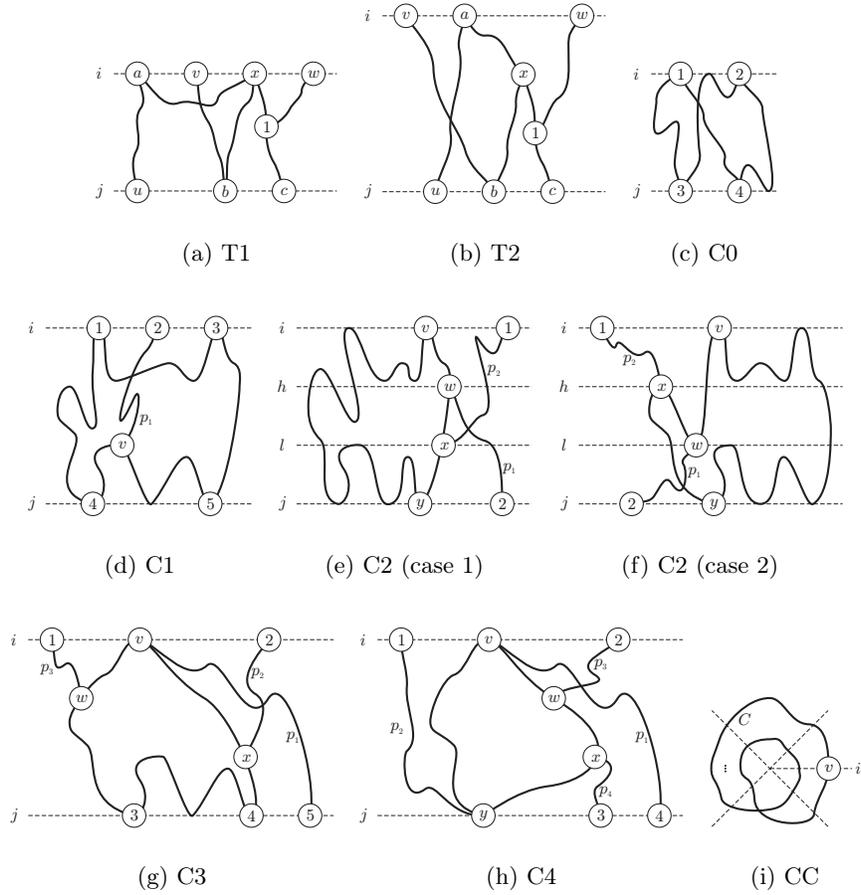
## 3 Cyclic Level Non-planarity Patterns

In this section we give a characterization of cyclic level non-planarity patterns in strongly connected graphs (SCLNP).

**Definition 1.** A (cyclic) level non-planarity pattern  $P$  is a set of (cyclic) level non-planar graphs with structural similarities. We call a pattern minimal if for each element of  $P$  the removal of one edge makes the graph (cyclic) level planar. A (cyclic) level graph  $G$  matches a pattern  $P$  if there exists  $p \in P$  s. t.  $p$  is a subgraph of  $G$ . A set of patterns  $S$  is minimal if each pattern in  $S$  is minimal.  $S$  is complete if each (cyclic) level non-planar graph matches a pattern in  $S$ .

Concerning level non-planarity patterns, Di Battista and Nardelli [4] presented three patterns for hierarchies (HLNP). This set is complete but not minimal. Healy et al. [7] gave seven minimal level non-planarity patterns for hierarchies (MHLNP). This complete set consists of two tree patterns T1 (Fig. 3(a)) and T2 (Fig. 3(b)), a level non-planar cycle C0 (Fig. 3(c)), and four level planar cycles with one (C1, Fig. 3(d)) to four (C4, Fig. 3(h)) paths starting from the cycle (C2 having two subcases). For the formal definition of the patterns see [7]. Note, that contrary to graph planarity, the partition of all minimal (cyclic) non-level planar graphs into patterns is somewhat arbitrary. Thus, nobody has treated the minimal cardinality of a minimal set of patterns.

Fowler and Kobourov [5] showed that the MHLNP set is not complete for general level graphs and added two more tree patterns, which are not needed for hierarchies, however. These nine patterns (MLNP) are minimal and complete for the general case. To formally describe cyclic level non-planarity patterns, we need the following definition.



**Fig. 3.** The cyclic level non-planarity patterns for strongly connected graphs (SCLNP)

**Definition 2.** Let  $G$  be a cyclic  $k$ -level graph. Let  $c \in \mathbb{N}$  and  $l \in \{1, \dots, k\}$ . Suppose that no edge crosses level  $l$  (if such an edge  $e = (u_G, v_G)$  exists, add a node  $d_G$  to level  $l$ , remove the edge  $e$  and add the edges  $(u_G, d_G)$  and  $(d_G, v_G)$  to the graph). The  $(c, l)$ -hierarchy  $H$  of  $G$  is a  $(ck + 1)$ -level hierarchy. For each node  $v_G$  on level  $l$   $H$  has  $c + 1$  duplicates  $v_1, v_{k+1}, \dots, v_{ck+1}$  with  $v_i$  on level  $i$ . Let  $w_G$  be a node with  $\phi_G(w_G) \neq l$  and let  $l' = \phi_G(w_G) - l + 1$  if  $\phi_G(w_G) > l$  and  $l' = \phi_G(w_G) - l + k + 1$  otherwise. For each such node  $w_G$ ,  $H$  has  $c$  duplicated nodes  $w_{l'}, w_{k+l'}, \dots, w_{(c-1)k+l'}$  with  $w_i$  on level  $i$ . For each edge  $e = (u_G, v_G)$  and for each duplicate  $u_i$  in  $H$  with  $i < ck + 1$   $H$  contains the edge  $(u_i, v_{i+\text{span}(e)})$ .

Informally speaking, the  $(c, l)$ -hierarchy  $H$  of  $G$  is obtained by splitting  $G$  at level  $l$  (thus creating a level graph) and duplicating the graph  $c$  times one below the other. We will use the notation  $v_i$  for a duplicate of  $v_G \in V(G)$  on level  $i$  in

$H$  in the following. Figure 1 shows a cyclic 5-level graph and Fig. 2 a level plane drawing of the corresponding  $(2, 1)$ -hierarchy. Note that a  $(c, l)$ -hierarchy does not include an embedding.

In a hierarchy, each node  $v$  with  $\phi(v) > 1$  has an ingoing edge. In strongly connected cyclic level graphs each node has an ingoing and outgoing edge. Therefore, it is not obvious that each of the seven MHLNP patterns can occur in the cyclic case. But for each of the patterns a strongly connected cyclic level graph can be constructed. Figures 4(a) to (h) show  $(1, 1)$ -hierarchies of strongly connected cyclic  $k$ -level graphs which match exactly one of the MHLNP patterns. The edges needed for the patterns are drawn as full lines. Obviously each of the MHLNP patterns is a proof of non-planarity in the cyclic case as well. Therefore, we define a cyclic version of each of the seven MHLNP patterns:

**Definition 3.** *Let  $G$  be a cyclic  $k$ -level graph. We say that  $G$  matches the pattern  $CT1$  if there exists  $c \in \mathbb{N}$  s. t. the  $(c, 1)$ -hierarchy of  $G$  matches  $T1$ . We define the remaining six patterns in an analog way and set  $SCLNP' = \{CT1, CT2, CC0, CC1, CC2, CC3, CC4\}$ .*

**Definition 4.** *Let  $G$  be a cyclic  $k$ -level graph and  $c \in \mathbb{N} \setminus \{0\}$ . A  $c$ -cycle is a simple cycle with span  $c \cdot k$ .*

Figure 4(i) shows the  $(1, 1)$ -hierarchy of a strongly connected cyclic 2-level graph which is a 2-cycle. Note that this graph is cyclic level non-planar, but does not match any pattern of  $SCLNP'$ . On the other hand none of the graphs in Fig. 4(a) to (h) contains a 2-cycle. Therefore, an eighth cyclic level non-planarity pattern is needed.

**Definition 5.** *We define  $CC$  as the set of all  $c$ -cycles in cyclic  $k$ -level graphs with  $c, k \in \mathbb{N}, c > 1$  (Fig. 3(i)). We set  $SCLNP = SCLNP' \cup \{CC\}$ .*

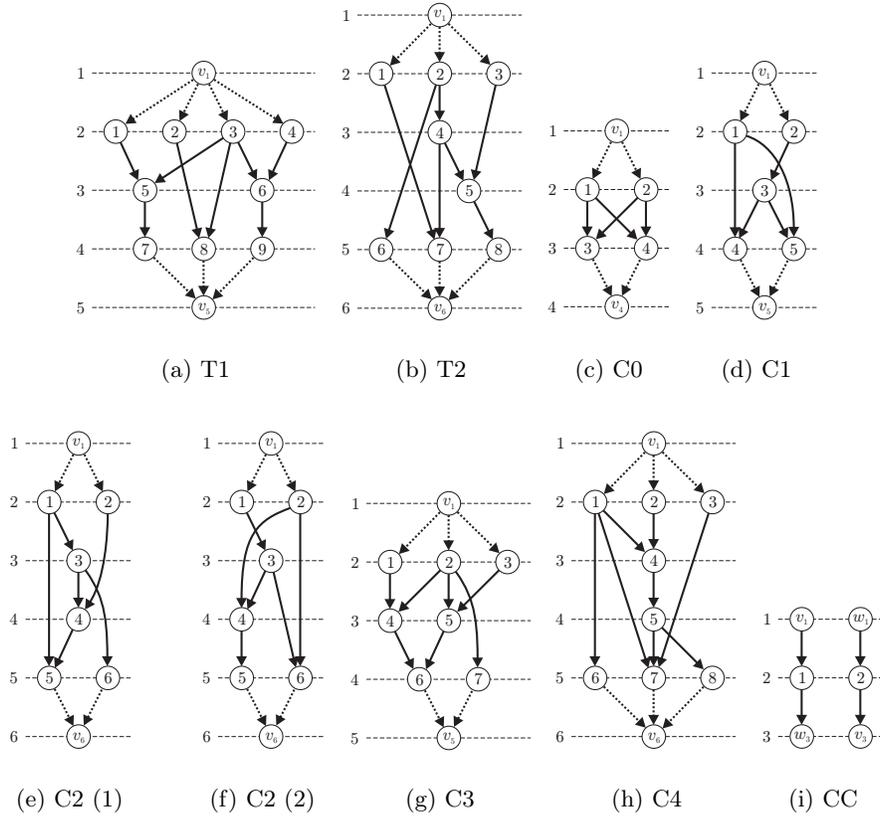
**Proposition 1.** *Let  $G$  be a cyclic  $k$ -level graph. If  $G$  matches  $CC$ , then  $G$  is cyclic level non-planar.*

Note that  $CC$  is a minimal pattern, as removing one edge from the cycle leads to a cyclic level planar graph.

**Definition 6.** *We call a  $(c, l)$ -hierarchy  $H$  of a cyclic level graph  $G$  strongly level planar if it is level planar and has a level planar embedding s. t. the first and last level have the same permutation. We call such an embedding a strongly level planar embedding.*

**Proposition 2.** *Let  $G$  be a strongly connected cyclic  $k$ -level graph and  $l \in \{1, \dots, k\}$ .  $G$  is cyclic level planar if and only if the  $(1, l)$ -hierarchy of  $G$  is strongly level planar. Let  $c \in \mathbb{N}$ . If the  $(c, l)$ -hierarchy of  $G$  is (strongly) level non-planar, then  $G$  is cyclic level non-planar.*

**Definition 7.** *Let  $H$  be a level graph with a fixed level planar embedding  $\mathcal{H}$  and  $u, v$  be two nodes on the same level. We say  $u < v$  ( $u > v$ ) if  $u$  lies left (right) of  $v$  in  $\mathcal{H}$ .*



**Fig. 4.** (1,1)-hierarchies of cyclic graphs matching exactly one of the SCLNP patterns

**Definition 8.** Let  $U, V, W$  be three permutations of the same node set. We define the lexicographical ordering with respect to  $U$  on the set of permutations in the following way: If  $V$  and  $W$  are the same permutations, then they are equal in the ordering. Otherwise there is a leftmost position on which  $V$  and  $W$  have different nodes. Let  $v$  and  $w$  be the nodes on this position in  $V$  and  $W$ , respectively. We define  $V < W$  if  $v < w$  in  $U$  and  $V > W$  if  $v > w$  in  $U$ .

Another way to look at this ordering is as follows: The permutation  $U$  defines an ordering on an alphabet. A permutation  $V$  is then smaller than  $W$  if the word it builds is smaller than the word of  $W$  in the lexicographical ordering.

**Lemma 1.** Let  $G$  be a strongly connected cyclic  $k$ -level graph s. t.  $G$  does not match  $CC$ . Let  $H$  be the  $(2,1)$ -hierarchy of  $G$ . Let  $H$  be level planar with a fixed level planar embedding  $\mathcal{H}$ . Then  $G$  is cyclic level planar and a cyclic level planar embedding  $\mathcal{G}$  of  $G$  exists s. t. the permutation of level 1 in  $\mathcal{G}$  is the same as the permutation of level  $k + 1$  in  $\mathcal{H}$ .

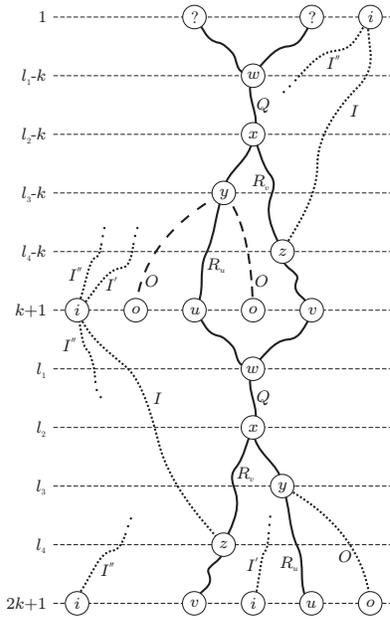


Fig. 5. Sketch for the proof of Lemma 1

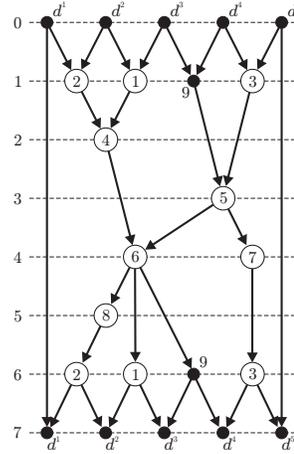


Fig. 6. Rigid (1, 1)-hierarchy

*Proof.* Assume for contradiction that such an embedding  $\mathcal{G}$  of  $G$  does not exist. Consider all level planar embeddings of  $H$  which have the same permutation of level  $k + 1$  as  $\mathcal{H}$ . None of these embeddings has this permutation on level  $2k + 1$ , too (otherwise  $G$  would be cyclic level planar with the same permutation on level 1). Of all these embeddings choose one which has the minimal permutation on level  $2k + 1$  in the lexicographical ordering with respect to the ordering of level  $k + 1$ . We consider this embedding from now on. We show that we can construct a new embedding with an even smaller permutation on level  $2k + 1$ , which is a contradiction.

As the permutations of level  $k + 1$  and  $2k + 1$  are not the same, there have to exist two nodes  $u$  and  $v$  s. t.  $u_{k+1} < v_{k+1}$  but  $u_{2k+1} > v_{2k+1}$ . W.l.o.g. let  $u_{2k+1}$  and  $v_{2k+1}$  be a pair of nodes with the wrong orientation that have the maximal number of nodes between them. See Fig. 5 for a sketch in which we omit the indices indicating the levels.

As  $G$  is strongly connected but does not match CC, each node lies on a 1-cycle. Thus, there have to be paths  $Q_u$  from  $u_{k+1}$  to  $u_{2k+1}$  and  $Q_v$  from  $v_{k+1}$  to  $v_{2k+1}$ . As the embedding is level planar,  $Q_u$  and  $Q_v$  cannot be disjoint. Therefore, we have a path  $Q_l$  from  $u_{k+1}$  to  $v_{2k+1}$  and a path  $Q_r$  from  $v_{k+1}$  to  $u_{2k+1}$ . As  $G$  does not match CC,  $Q_l$  and  $Q_r$  cannot be disjoint. Even more than that, there has to exist a node which lies on all paths from  $u_{k+1}$  to  $v_{2k+1}$  and on all paths from  $v_{k+1}$  to  $u_{2k+1}$  (otherwise the leftmost path from  $u_{k+1}$  to  $v_{2k+1}$  and the rightmost path from  $v_{k+1}$  to  $u_{2k+1}$  would be disjoint and generate a graph in

CC). Let  $w_{l_1}$  and  $x_{l_2}$  be the uppermost and lowest such nodes, respectively, and  $Q$  be one path between them.

Let  $R_v$  be the leftmost path from  $x_{l_2}$  to  $v_{2k+1}$ . Let  $R_u$  be the rightmost path from  $x_{l_2}$  to  $u_{2k+1}$ . We now prove that we can flip all nodes between  $R_v$  and  $R_u$ , thus creating a smaller permutation on level  $2k + 1$  in the lexicographical ordering and thus a contradiction.

Consider the nodes on the path  $R_u$  from level  $l_2 + 1$  to  $2k$ . Assume for contradiction that a node  $y_{l_3}$  on the path has an outgoing edge right to  $R_u$ . We follow this path  $O$  downwards. If  $O$  ends on  $R_u$ , then  $R_u$  was not the rightmost path. Therefore, it has to reach a node  $o_{2k+1}$  on level  $2k+1$  right of  $u_{2k+1}$ . As  $O$  is disjoint to  $R_v$ ,  $o_{k+1}$  has to lie left of  $v_{k+1}$  (we do not know whether  $o_{k+1} < u_{k+1}$  or  $o_{k+1} > u_{k+1}$  holds). But then  $o_{2k+1}$  and  $v_{2k+1}$  have the wrong orientation and have more nodes between them than  $v_{2k+1}$  and  $u_{2k+1}$ . A contradiction. The same argument can be used for  $R_v$  (switching left and right).

Now consider the nodes on the path  $R_v$  from level  $l_2 + 1$  to  $2k + 1$ . Assume for contradiction that a node  $z_{l_4}$  on the path has an ingoing edge left to  $R_v$ . We follow this path  $I$  upwards. If  $I$  ends on  $R_v$  below or on  $x_{l_2}$ , then  $R_v$  was not the leftmost path. If  $I$  ends on  $Q \setminus \{x_{l_2}\}$ , then  $x_{l_2}$  would not lie on each path from  $u_{k+1}$  to  $v_{2k+1}$ . If it ends on the leftmost connection of  $u_{k+1}$  to  $w_{l_1}$  above  $w_{l_1}$ , then there would be disjoint paths from  $u_{k+1}$  to  $v_{2k+1}$  and from  $v_{k+1}$  to  $u_{2k+1}$ . The only remaining possibility is that  $I$  reaches the level  $k + 1$  on a node  $i_{k+1}$  left of  $u_{k+1}$ . Due to the path  $I$ ,  $i_1$  has to lie right of  $u_1$  and  $v_1$ . However, we do not know whether  $u_1 < v_1$  or  $u_1 > v_1$  holds.

We now consider the position of  $i_{2k+1}$ . If  $i_{2k+1} > u_{2k+1}$  and, therefore,  $i_{2k+1} > v_{2k+1}$ , then  $i_{2k+1}$  and  $v_{2k+1}$  have the wrong orientation and more nodes between them than  $u_{2k+1}$  and  $v_{2k+1}$ . If  $i_{2k+1}$  lies between  $v_{2k+1}$  and  $u_{2k+1}$ , we consider the path  $I'$  from  $i_{2k+1}$  upwards. If  $I'$  reaches  $R_u$ , then we have disjoint paths from  $v_{k+1}$  to  $i_{2k+1}$  and from  $i_{k+1}$  to  $v_{2k+1}$  and thus  $G$  matches CC. If  $I'$  reaches  $R_v$  first, then  $I'$  would cause a crossing from  $i_{k+1}$  upwards. The remaining possibility is that  $i_{2k+1} < v_{2k+1} < u_{2k+1}$ . Now the same path  $I''$  from  $i_1$  to  $i_{k+1}$  and from  $i_{k+1}$  to  $i_{2k+1}$  has to exist.  $I''$  cannot be disjoint with  $Q_u$  or  $Q_v$  as  $i_1$  is right of  $u_1$  and  $v_1$  but  $i_{k+1}$  is left of  $u_{k+1}$  and  $v_{k+1}$ . Therefore, from  $i_{2k+1}$  upwards  $I''$  has to reach  $I$  or  $R_v$  first to reach  $Q_u$ . In both cases a crossing from  $i_{k+1}$  upwards occurs, which is a contradiction. The same argument can be used to show that no path from  $R_u$  upwards exists.

As a consequence, we do not have any outgoing or ingoing edges on  $R_v$  to the left between  $l_2 + 1$  and  $2k + 1$ . Analogously, we do not have outgoing or ingoing edges to the right of  $R_u$  between the same levels. So we can flip the subgraph between  $R_v$  and  $R_u$  and thus create a permutation of level  $2k+1$  which is smaller than the given one in the lexicographical ordering with respect to the ordering of level  $k + 1$ . A contradiction. Thus,  $G$  is cyclic level planar with an embedding  $\mathcal{G}$  s. t. the permutation of level 1 in  $\mathcal{G}$  is the permutation of level  $k + 1$  in  $\mathcal{H}$ .  $\square$

Figure 1 shows a strongly connected cyclic 5-level graph and Fig. 2 an arbitrary level plane drawing of its  $(2,1)$ -hierarchy. Note that levels 1, 6, and 11 have three different permutations. According to Lemma 1, we can fix the embedding

of level 6 and change the permutation of level 11 to the permutation of level 6. We search for two nodes on level 11 which have the wrong orientation according to level 6 and the maximal number of nodes between them. These nodes are  $u = 1$  and  $v = 9$ . We get  $w = x = 6$  and flip the tree below node 6. After that 1 and 2 have the wrong orientation (with  $w = 4$  and  $x = 6$ ) and we flip these two nodes. Thereafter, levels 6 and 11 have the same permutation. This permutation is used for the cyclic level plane drawings in Fig. 1.

**Theorem 1.** *Let  $G$  be a strongly connected cyclic  $k$ -level graph.  $G$  is cyclic level planar if and only if it does not match a pattern in SCLNP.*

*Proof.* “ $\Rightarrow$ ” We show the contrapositive. If  $G$  matches a pattern of SCLNP’, then there exists  $c \in \mathbb{N}$  s. t. the  $(c, 1)$ -hierarchy  $H$  of  $G$  matches an MHLNP pattern. Therefore,  $H$  is level non-planar. According to Proposition 2,  $G$  is cyclic level non-planar then. If  $G$  matches the pattern CC, then  $G$  is cyclic level non-planar according to Proposition 1.

“ $\Leftarrow$ ” We show the contrapositive. Let  $G$  be cyclic level non-planar and let  $H$  be its  $(2, 1)$ -hierarchy. If  $H$  is level non-planar, then  $H$  matches a MHLNP pattern and, therefore,  $G$  matches a SCLNP’ pattern. If  $H$  is level planar, then (the contrapositive of) Lemma 1 shows that  $G$  matches the pattern CC.  $\square$

Note that according to Lemma 1 for each strongly connected cyclic  $k$ -level non-planar graph not matching CC its  $(2, 1)$ -hierarchy matches an MHLNP pattern. Therefore, patterns in SCLNP’ can be limited to  $2k + 1$  levels.

## 4 Cyclic Level Planarity Testing and Embedding

In this section we give a simple linear time level planarity testing and embedding algorithm.

**Definition 9.** *Let  $G$  be a cyclic  $k$ -level graph and  $H$  the  $(2, 1)$ -hierarchy of  $G$ . Let  $H$  be level planar with embedding  $\mathcal{H}$ . Let  $F = (v_{k+1}^1, v_{k+1}^2, \dots, v_{k+1}^s)$  be the permutation of level  $k+1$  in  $\mathcal{H}$ . The rigid  $(1, 1)$ -hierarchy  $H'$  of  $\mathcal{H}$  consists of the  $(1, 1)$ -hierarchy of  $G$  and the additional levels 0 and  $k+2$ . Level 0 has the nodes  $d_0^1, d_0^2, \dots, d_0^{s+1}$  and level  $k+2$  the nodes  $d_{k+2}^1, d_{k+2}^2, \dots, d_{k+2}^{s+1}$ .  $H'$  contains the edges  $(d_0^i, v_1^i)$ ,  $(d_0^{i+1}, v_1^i)$ ,  $(v_{k+1}^i, d_{k+2}^i)$  and  $(v_{k+1}^i, d_{k+2}^{i+1})$  for each  $i \in \{1, \dots, s\}$  as well as the edges  $(d_0^1, d_{k+2}^1)$  and  $(d_0^{s+1}, d_{k+2}^{s+1})$ .*

Note that the rigid  $(1, 1)$ -hierarchy  $H'$  of  $\mathcal{H}$  is level planar if and only if it has an embedding s. t. the levels 1 and  $k+1$  have the same permutation  $F$ . Then  $H$  is strongly level planar and  $G$  cyclic level planar. From Theorem 1 we get the following idea for a cyclic level planarity testing and embedding algorithm.

Let  $G = (V(G), E(G), \phi_G)$  be a strongly connected cyclic  $k$ -level graph. We first test whether  $|E(G)| \leq 3|V(G)| - 6$  holds (otherwise  $G$  cannot be (cyclic level) planar (Euler)). Construct the  $(2, 1)$ -hierarchy  $H$  of  $G$  then. If  $H$  is level non-planar, then  $G$  is cyclic level non-planar. Otherwise let  $\mathcal{H}$  be a level planar

embedding of  $H$ . We construct the rigid  $(1,1)$ -hierarchy  $H'$  of  $\mathcal{H}$  and test its level planarity. If it fails  $G$  is cyclic level non-planar. If it does not fail, we transform the level planar embedding  $\mathcal{H}'$  of  $H'$  into a cyclic level planar embedding  $\mathcal{G}$  of  $G$  in a straightforward way: Let  $v_G \in V(G)$  with  $\phi_G(v_G) \neq 1$ . Let  $v_l$  be the corresponding node in  $H'$ . We set  $N_G^-(v_G) = N_{H'}^-(v_l)$  and  $N_G^+(v_G) = N_{H'}^+(v_l)$ . For a node  $v_G \in V(G)$  with  $\phi_G(v_G) = 1$  we set  $N_G^-(v_G) = N_{H'}^-(v_{k+1})$  and  $N_G^+(v_G) = N_{H'}^+(v_1)$ . In both cases we identify the nodes in  $H'$  with the corresponding nodes in  $G$ .

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**Algorithm 1:** cyclicLevelPlanarEmbedding
 

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**Input:** A strongly connected cyclic  $k$ -level graph  $G = (V(G), E(G), \phi_G)$   
**Output:** A cyclic level planar embedding  $\mathcal{G}$  or *false*

- 1 **if**  $|E(G)| > 3|V(G)| - 6$  **then**
- 2    **return false**
- 3 Let  $H$  be the  $(2,1)$ -hierarchy of  $G$
- 4 **if**  $\neg \text{levelPlanar}(H)$  **then**
- 5    **return false**
- 6 Let  $\mathcal{H}$  be a level planar embedding of  $H$
- 7 Let  $H'$  be the rigid  $(1,1)$ -hierarchy of  $\mathcal{H}$
- 8 **if**  $\neg \text{levelPlanar}(H')$  **then**
- 9    **return false**
- 10 Let  $\mathcal{H}'$  be a level planar embedding of  $H'$
- 11 Construct cyclic level planar embedding  $\mathcal{G}$  of  $G$  from  $\mathcal{H}'$
- 12 **return**  $\mathcal{G}$

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**Theorem 2.** *Cyclic level planarity testing and embedding on strongly connected cyclic level graphs can be achieved by Algorithm 1 in linear time.*

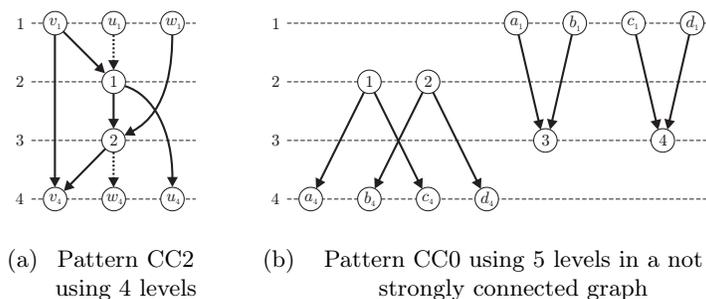
*Proof.* The correctness of Algorithm 1 follows directly from Theorem 1. To prove its time complexity, we consider the construction of the  $(2,1)$ -hierarchy  $H$  of  $G$  first. The addition of dummy nodes on level 1 increases the number of nodes and edges by at most  $E(G)$ . After that the graph is duplicated: Each node on level 1 has three duplicates, all remaining nodes and all edges have two duplicates. Therefore, the size of  $H$  is linear in the size of  $G$ . All steps can easily be done in linear time. To test the level planarity of  $H$  any linear time level planarity testing and embedding algorithm for hierarchies can be used [4,9]. The construction of the  $(1,1)$ -hierarchy is possible in linear time as well. Let  $w$  be the number of nodes on level 1 in this hierarchy. To build the rigid  $(1,1)$ -hierarchy, we add  $2(w+1)$  nodes and  $4w+2$  edges and again use a linear time level planarity testing and embedding algorithm. The construction of  $\mathcal{G}$  from  $\mathcal{H}'$  can again easily be done in linear time.  $\square$

### 5 Summary and Open Problems

We have shown that each of the seven MHLNP patterns of Healy et al. [7] are necessary in the strongly connected cyclic level case. To build a complete set, an eighth pattern is needed: a simple cycle wrapping around the center more than once (CC). This leads to a simple linear time level planarity testing and embedding algorithm.

As open problems remain finding a linear time planarity testing and embedding algorithm for arbitrary cyclic level graphs, a stronger characterization of the SCLNP patterns as well as finding patterns for the arbitrary cyclic level case.

*Conjecture 1.* Let  $G$  be a strongly connected cyclic  $k$ -level graph.  $G$  is cyclic level planar if and only if  $G$  does not match a pattern in SCLNP with the patterns in SCLNP' using at most  $k + 1$  levels.



**Fig. 7.** (1,1)-hierarchies of cyclic 3-level graphs matching one SCLNP pattern

Figure 7(a) shows a (1,1)-hierarchy of a cyclic 3-level graph which matches the pattern CC2 (case 1) and uses 4 levels. If Conjecture 1 holds, it is not possible to construct a strongly connected cyclic  $k$ -level non-planar graph not matching CC s. t. all patterns use more than  $k + 1$  levels. Obviously, this is false for not strongly connected graphs, as the cyclic 3-level graph in Fig. 7(b) shows. The only pattern it matches is the pattern CC0 using 5 levels. This example can easily be enlarged s. t. the pattern uses an arbitrary amount of levels.

*Conjecture 2.* Let  $G$  be a (not necessarily strongly connected) cyclic  $k$ -level graph.  $G$  is cyclic level planar if and only if there does not exist  $c \in \mathbb{N}$  s. t. the  $(c, 1)$ -hierarchy matches one of the nine MLNP patterns and  $G$  does not contain an undirected simple cycle wrapping around the center more than once.

Lemma 2 is a strong indication for Conjecture 1:

**Lemma 2.** *Let  $G$  be a strongly connected cyclic  $k$ -level graph not matching CC. If  $G$  matches an SCLNP' pattern using less than  $2k + 1$  and more than  $k + 2$  levels, then it matches another instance of an SCLNP' pattern.*

*Proof.* Let  $H$  be the  $(2, 1)$ -hierarchy of  $G$ . W.l.o.g. let 1 be the first level of the pattern in  $H$ . Note that  $H$  matches the pattern, but the  $(2, 2)$ -hierarchy  $H'$  of  $G$  does not match it, as the level 1 of  $G$  is missing at the top. As more than  $k + 2$  levels are used by the pattern, the pattern is not completely there from level  $k$  downwards as well.  $H'$  has to match another SCLNP' pattern, as we could use  $H'$  instead of  $H$  in Lemma 1.  $\square$

Note that Lemma 2 does not make a statement on patterns using  $2k + 1$  or  $k + 2$  levels. Nevertheless, we conjecture that the patterns use  $k + 1$  levels at most, as all strongly connected cyclic level graphs matching a longer pattern seem to match a shorter pattern or a CC pattern as well.

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