Classification of Planar Upward Embedding

Christopher Auer, Christian Bachmaier,
Franz J. Brandenburg, and Andreas Gleißner

Department of Informatics and Mathematics, University of Passau
{auerc,brandenb,bachmaier,gleissner}@fim.uni-passau.de

UNIVERSITÄT PASSAU
Fakultät für Informatik und Mathematik

Technical Report, Number MIP-1106
Department of Informatics and Mathematics
University of Passau, Germany
September 2011

Classification of Planar Upward Embedding*

Technical Report MIP-1106
September, 2011

Christopher Auer, Christian Bachmaier,
Franz J. Brandenburg, and Andreas Gleißner

University of Passau, 94030 Passau, Germany,
{auerc,bachmaier,brandenb,gleissner}@fim.uni-passau.de

Abstract. We consider planar upward drawings of directed graphs on arbitrary surfaces where the upward direction is defined by a vector field. This generalizes earlier approaches using surfaces with a fixed embedding in $\mathbb{R}^3$ and introduces new classes of planar upward drawable graphs, where some of them even allow cycles. Our approach leads to a classification of planar upward embeddability.

In particular, we show the coincidence of the classes of planar upward drawable graphs on the sphere and on the standing cylinder. These classes coincide with the classes of planar upward drawable graphs with a homogeneous field on a cylinder and with a radial field in the plane.

A cyclic field in the plane introduces the new class **RUP** of upward drawable graphs, which can be embedded on a rolling cylinder. We establish strict inclusions for planar upward drawability on the plane, the sphere, the rolling cylinder, and the torus, even for acyclic graphs. Finally, upward drawability remains **NP**-hard for the standing cylinder and the torus; for the cylinder this was left as an open problem by Limaye et al.

1 Introduction

Directed graphs are often used as a model for structural relations where the edges express dependencies. Such graphs are often acyclic and are drawn as hierarchies using the hierarchical approach introduced by Sugiyama et al. [22]. This drawing style transforms the edge direction into a geometric direction: all edges point upward. A graph is upward planar, for short **UP**, if it can be embedded into the plane such that the curves of the edges are monotonically increasing in y-direction with no crossing edges. **UP** is well-understood; see the comprehensive study in [5]. A graph is upward planar if and only if it is a subgraph of a planar st-graph. The graphs from **UP** admit straight-line upward drawings, which may require an area of exponential size, or upward polyline drawings on quadratic area using $O(n)$ many bends. An important result of Garg and Tamassia [10] states the **NP**-completeness of the recognition problem: Is a directed graph in **UP**? On the

* Supported by the Deutsche Forschungsgemeinschaft (DFG), grant Br835/15-1.

other hand, there are efficient polynomial time algorithms for upward planarity tests, if the graphs are given with an embedding or have a single source or are triconnected.

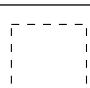
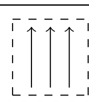
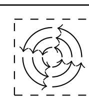
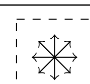
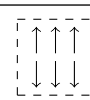
There were some approaches to generalize upward planarity on other surfaces using a fixed embedding of the surface in $\mathbb{R}^3$. Thomassen [23] studied graphs with a single source and a single sink on a standing cylinder. Foldes et al. [9] investigated ordered sets on the sphere and on a cylinder as a truncated sphere, and Hashemi et al. [7, 12, 13] generalized results on planarity from the plane to the sphere, including the **NP**-hardness of the recognition problem. They characterized the graphs with a spherical upward drawing as the subgraphs of the directed planar graphs with one source and one sink. Thus upward planarity and upward sphericity are distinguished by the *st*-edge connecting the single source and the single sink in the planar case. Dolati et al. [6, 8] studied upward planarity on the lying and the standing torus, and Mohar and Rosenstiehl [19] characterize toroidal maps with an upward orientation.

Planar upward drawings on the cylinder were also addressed from the viewpoint of the circuit value problem (CVP) [11, 16, 24]. In these papers the above papers were overseen, and the **NP**-hardness of upward cylindricality is stated as an open problem [16]. We solve this by using the **NP**-hardness for upward spherical and the coincidence of spherical and cylindrical upward planarity established in this paper.

In our approach we use the model of the fundamental polygon to define surfaces such as the plane, the cylinder and the torus. The plane is identified with the manifold $I \times I$, where I is the open interval from -1 to $+1$. The standing and rolling cylinder are obtained by identifying a pair of opposite sides, and the torus by a simultaneous identification of both pairs of opposite sides.

Upwardness is defined by a vector field and gives rise to the common (*strict*) increasing and the *weak* non-decreasing case. A vector field assigns a two-dimensional vector to each point (x, y) indicating the direction of the field. The basic case is the *null field* N, which assigns the null vector $(0, 0)$ everywhere. Then an upward direction becomes vacuous, and weakly upward planar coincides with planar. The *homogeneous field* H assigns the direction $(0, 1)$ and thus describes upward in y-dimension as it is commonly used. In addition, we use the *cyclic*, *radial* and *antiparallel fields* C, R and A, see Table 1.

Table 1. Typical fields

null	homogeneous	cyclic	radial	antiparallel
				
$(x, y) \mapsto (0, 0)$	$(x, y) \mapsto (0, 1)$	$(x, y) \mapsto (-y, x)$	$(x, y) \mapsto (x, y)$	$(x, y) \mapsto (0, \sin(y\pi))$

We introduce a new class of planar upward drawings on the rolling cylinder which is called **RUP**. Graphs of **RUP** may have cycles. It turns out that the rolling cylinder is stronger than the standing cylinder even for acyclic graphs. The graphs of **RUP** are related to planar recurrent hierarchies, which were introduced by Sugiyama et al. [22] as a cyclic version of their hierarchical approach and were recently studied in [3]. In recurrent hierarchies the levels are numbered from 0 to $k-1$. The edges are upward where the difference of the levels of the vertices is computed modulo k. Hence, all cycles are unidirectional.

Another subclass of **RUP** are the graphs with a queue layout, see [1]. The input-output behavior of a queue is represented by a graph such that the behavior is legal if and only if the graph has a **RUP** embedding with all vertices placed on a horizontal line.

Our contributions are a general approach towards planar upward embeddings (Sect. 2). In Sect. 3 we unify the concepts on the sphere and establish a hierarchy for the plane, sphere, rolling cylinder and torus. Finally, the **NP**-hardness of the recognition problem is addressed. We conclude with an outlook on future work.

2 Upward Embeddings with Vector Fields on Surfaces

Let $G = (V, E)$ be a simple directed graph with a finite set of vertices V and a finite set of directed edges E. A *surface* $\mathbb{S}$ is a two-dimensional differentiable manifold [17, 20]. An open interval from a to b is denoted by $]a, b[$ and a closed interval by $[a, b]$. $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^2$. For a map $f : A \to B$ and a subset $A' \subseteq A$ denote the image of A' under f by $f[A']$. For any point $p = (p_1, p_2)$ define $x(p) = p_1$ and $y(p) = p_2$.

A *drawing* $\Gamma(G)$ on $\mathbb{S}$ is a mapping where each vertex $v \in V$ is mapped to a unique point $\Gamma(v) \in \mathbb{S}$, and each edge $(u, v) \in E$ is mapped to a piecewise continuously differentiable curve $\Gamma(u, v) : [0, 1] \to \mathbb{S}$ which starts at u and ends at v and is disjoint to the other vertex points. $\Gamma(u, v)$ does not self-intersect. When it is clear from the context, we say that $v \in V$ is *placed* at $\Gamma(v)$ and we do not distinguish between an edge $e \in E$ and its curve $\Gamma(e)$. Additionally, Γ stands for the set of points in the drawing.

Two edges $e_1 \neq e_2 \in E$ *cross* if they have a common point apart from a common endpoint. $\Gamma(G)$ is called a *plane* drawing if it is crossing-free. *Strict upward planarity* asks if a given graph admits a plane drawing where all edges are drawn monotonically increasing in a common upward direction. In the *weak* version the edges may be drawn monotonically non-decreasing. It is well-known that this makes no difference on the plane.

As outlined in Sect. 1 most prior attempts towards planar upward embeddings on the sphere, the cylinder, or the torus use a fixed embedding of the surface in $\mathbb{R}^3$ and define upward in y-direction [6–8, 11, 13, 16]. They describe the sphere and the (standing) cylinder by Cartesian coordinates $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and $\{(x, y, z) : x^2 + z^2 = 1, -1 \leq y \leq 1)\}$, respectively. These classes are called *spherical* and *cylindrical*. An alternative approach was used by Mohar, Rosenstiehl and Thomassen [19, 23] embedding graphs on the *flat torus* represented by its fundamental polygon. We generalize the idea by utilizing vector fields, i. e., a drawing is upward if all edge curves "go with the flow".

More formally, let $F : \mathbb{S} \to \mathbb{R}^2$ be a vector field on $\mathbb{S}$. Let $Cr(p) \subsetneq [0,1]$ be the preimage of the bends of the curve p. $Cr(p)$ is the countable critical point set of a piecewise continuously differentiable curve $p : [0,1] \to \mathbb{S}$. We say that p (*weakly*) *respects* F if

$$\underset{t \in [0,1] \setminus Cr(p)}{\forall} \langle p'(t), F(p(t)) \rangle > 0 \quad (\text{resp.} \geq 0), \tag{1}$$

where p' is the first order derivative of p. Likewise, a drawing Γ (*weakly*) *respects* F if $\Gamma(e)$ (weakly) respects F for each edge $e \in E$. Then at each point of a directed edge the angle between its tangent vector and the vector field is less (not more) than $\frac{\pi}{2}$. We call a graph *(weakly) upward embeddable on* $\mathbb{S}$ *in respect to* F if it admits a plane drawing (weakly) respecting F. We say that G *is a drawn (weakly) upward on* $(\mathbb{S}, F)$. Note that (1) holds true independently of the norm of $F(\cdot)$, i. e., only its direction is relevant.

The general definition allows for a plethora of combinations of surfaces and vector fields. From a graph-theoretic point of view many of them are equivalent in respect to upward embeddability. For reducing redundancy we consider mappings between surfaces which shall preserve the upward embeddability and so obtain equivalences.

Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be smooth manifolds, i. e., locally similar to a linear space, with vector fields F_1 and F_2, respectively. Let $f : \mathbb{S}_1 \to \mathbb{S}_2$ be an injective smooth mapping between the surfaces. In the following we derive a way to express whether or not f also somehow "maps F_1 to F_2". The technique is also known as the *pushforward* of f [14]. Let z be any point in $\mathbb{S}_1$ and $p : [0,1] \to \mathbb{S}_1$ be a smooth curve (not necessarily representing an edge) tangent to F_1 in z, i. e., $p(0) = z$ and $p'(0) = F_1(z)$. We derive how f acts on $F_1(z)$ by considering the derivative of $f(p)$ at 0,

$$(f \circ p)'(0) = (f' \circ p)(0) \cdot p'(0) = f'(p(0)) \cdot p'(0) = (f'(p(0))) \cdot F_1(z) = f'(z) \cdot F_1(z)\,.$$

Due to the identification of the tangent space of $\mathbb{S}_1$ with $\mathbb{R}^2$ we can express $f'(z)$ by the Jacobian $J_f(z)$. From this we obtain the requirement for F_1 and F_2 of being *f-related* [14]: For each $z \in \mathbb{S}_1$, $J_f(z) \cdot F_1(z) = F_2(f(z))$, or equivalently, $F_2(z) = J_f(f^{-1}(z)) \cdot F_1(f^{-1}(z))$. As we are only interested in the direction of vectors rather than their lengths, denote by $u \simeq v$ if $u = cv$ for some positive real constant c. F_1 and F_2 are said to be *f-related up to normalization* if $F_2(z) \simeq J_f(f^{-1}(z)) \cdot F_1(f^{-1}(z))$ for each $z \in \mathbb{S}_1$. We introduce a second property to guarantee that upward embeddability is preserved.

Definition 1. *Let* $\mathbb{S}_1$ *and* $\mathbb{S}_2$ *be smooth manifolds with vector fields* F_1 *and* F_2*, respectively. We call a smooth injective homeomorphism* $f : \mathbb{S}_1 \to \mathbb{S}_2$ *to be* field preserving *from* $(\mathbb{S}_1, F_1)$ *to* $(\mathbb{S}_2, F_2)$ *if* F_1 *and* F_2 *are f-related up to normalization, and for any smooth curve* $p : [0,1] \to \mathbb{S}_1$*,*

$$\operatorname{sgn}\langle p'(0), (F_1 \circ p)(0) \rangle = \operatorname{sgn}\langle (f \circ p)'(0), (F_2 \circ f \circ p)(0)) \rangle\,.$$

Rephrasing the above, f preserves the (non-)acuteness of the angle between a tangent vector and the vector field at any point. This gives rise to the following proposition.

Proposition 1. *Let G be a simple directed graph and let $\mathbb{S}_1$ and $\mathbb{S}_2$ be differentiable two-dimensional manifolds with vector fields F_1 and F_2, respectively. Let $\mathbb{S}_1'$ be a subset of $\mathbb{S}_1$ such that in respect to F_1, any graph upward embeddable on $\mathbb{S}_1$ is also upward embeddable on $\mathbb{S}_1'$. If G is (weakly) upward embeddable on $\mathbb{S}_1$ in respect to F_1 and there is a field-preserving map f from $(\mathbb{S}_1', F_1)$ to $(\mathbb{S}_2, F_2)$, then G is also (weakly) upward embeddable on $\mathbb{S}_2$ in respect to F_2.*

Proof. Assume G is upward embeddable on $\mathbb{S}_1$ in respect to F_1. Let Γ be a plane drawing of G on $\mathbb{S}_1'$ respecting F_1. The drawing $f[\Gamma]$ of G on $\mathbb{S}_2$ is plane as f is differentiable. It also respects F_2 as f specifically preserves the acuteness of the angles between the vector field and the tangents of the edge curves. □

Note that the well-known conformal, i. e., angle-preserving, maps are just a special case of the field-preserving maps if they relate F_1 to F_2 up to normalization. Additionally, any composition of field-preserving maps is field-preserving in respect to the corresponding manifolds and vector fields.

We define $(\mathbb{S}_1, F_1) \sim (\mathbb{S}_2, F_2)$ if and only if there are functions f and g such that f is field-preserving from $(\mathbb{S}_1, F_1)$ to $(\mathbb{S}_2, F_2)$ and g is field-preserving from $(\mathbb{S}_2, F_2)$ to $(\mathbb{S}_1, F_1)$. Proposition 1 allows us to speak of *upward embeddability of G in the equivalence class* $[\mathbb{S}, F]$. We can define the directed simple graph classes

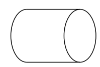
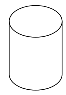
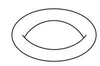
$$[\![\mathbb{S}F]\!]_{\mathrm{s}} = \{G : G \text{ is (strictly) upward embeddable on } [\mathbb{S}, F]\} \text{ and}$$
$$[\![\mathbb{S}F]\!]_{\mathrm{w}} = \{G : G \text{ is weakly upward embeddable on } [\mathbb{S}, F]\},$$

where the subscripts indicate the strict or weak case. This class scheme enables us to classify and generalize prior approaches of upward planarity. We restrict ourselves to manifolds which are obtained from a square where optionally opposite sides are identified. Thus any of the considered manifolds can be represented by rectangular fundamental polygons [18]. Let $I =]-1, 1[$ and derive $I_\circ$ from I by identifying its boundaries -1 and 1. With a slight abuse of language we define the following two-dimensional manifolds as the product manifolds of I and $I_\circ$ with their natural differentiable structure: The *plane* $\mathbb{P} = I \times I$, the *standing cylinder* $\mathbb{C}_s = I_\circ \times I$, the *rolling cylinder* $\mathbb{C}_r = I \times I_\circ$, and the *torus* $\mathbb{T} = I_\circ \times I_\circ$. See Table 2 for an illustration.

A point in each of the defined manifolds can be represented by a pair (x, y). A vector field assigns a two-dimensional vector to each such pair (x, y) that defines the direction of the field at (x, y). A basic case is the null field N, which assigns the null vector $(0, 0)$ everywhere. Then any direction of the edges weakly respects the null field. Therefore, the graphs $[\![\mathbb{P}N]\!]_{\mathrm{w}}$, i. e., upward embeddable in the plane and weakly respecting the null field, are exactly the planar graphs in the usual sense, denoted by $\mathbf{P}$. Similarily, $\mathbf{T} = [\![\mathbb{T}N]\!]_{\mathrm{w}}$ are the toroidal graphs.

Next we consider the homogeneous field H that maps each point to $(0, 1)$. Then the upward planar graphs $\mathbf{UP}$ are exactly captured by $[\![\mathbb{P}H]\!]_{\mathrm{s}}$. We additionally

Table 2. Surfaces resulting from the cross products of I and $I_\circ$

$\times$	$I =]-1,1[$	$I_\circ$
$I =]-1,1[$	$\mathbb{P}$	$\mathbb{C}_r$ 
$I_\circ$	$\mathbb{C}_s$ 	$\mathbb{T}$ 

investigate the following graph classes: **SUP** $= [\![\mathbb{C}_s H]\!]_\mathrm{s}$, **wSUP** $= [\![\mathbb{C}_s H]\!]_\mathrm{w}$, **RUP** $= [\![\mathbb{C}_r H]\!]_\mathrm{s}$, **wRUP** $= [\![\mathbb{C}_r H]\!]_\mathrm{w}$, **UT** $= [\![\mathbb{T} H]\!]_\mathrm{s}$, and **wUT** $= [\![\mathbb{T} H]\!]_\mathrm{w}$, which define (weakly) upward planarity on the standing and rolling cylinder, and on the torus, respectively.

3 Classification of Upward Drawings

First we show that planar upward drawings on the sphere, the standing cylinder and the plane with the radial field coincide both in the strict and in the weak versions. Instead of proving that the spherical and cylindrical graph classes are equal according to their graph-theoretical characterizations from [12, 15], our proof makes use of the definitions from Sect. 2 by transforming the surfaces with their endowed fields into each other.

Theorem 1. *For a graph G the following statements are equivalent.*

(i) $G \in$ ***SUP*** *(*$G \in$ ***wSUP****)*
(ii) G is (weakly) spherical
(iii) G is (weakly) cylindrical
(iv) $G \in [\![\mathbb{P}R]\!]_s$ *(*$G \in [\![\mathbb{P}R]\!]_w$*)*

Proof. All of the following arguments apply to the weak and the strict case. We first show *(ii)* $\Rightarrow$ *(i)*. Consider an upward drawing Γ of G on the sphere $\mathbb{S}_1$. First assume that there is no vertex placed on the poles, i. e., with coordinates $(0,1,0)$ or $(0,-1,0)$. Let $y_{\max}$ be the maximum y-coordinate of vertices of G. Note that there is no point of an edge above $y_{\max}$ as otherwise the upwardness is violated. Analogously define $y_{\min}$. Let $\mathbb{S}'_1 = \{(x,y,z) : x^2 + y^2 + z^2 = 1, y_{\min} < y < y_{\max}\}$, i. e., $\mathbb{S}'_1$ is the truncated sphere [9]. We use the angle-preserving Mercator projection M [21] to map $\mathbb{S}'_1$ to the rectangle $[x'_{\min}, x'_{\max}[\times]y'_{\min}, y'_{\max}[$ in the plane. Afterwards, we scale and translate $M[\Gamma]$ to obtain a drawing in the fundamental polygon $\mathbb{C}_s$ by

$$f : (x,y) \mapsto \left(\frac{2x}{x'_{\max} - x'_{\min}}, \frac{2y}{y'_{\max} - y'_{\min}}\right) + (\Delta_x, \Delta_y)\,, \tag{2}$$

where $\varDelta_x$ and $\varDelta_y$ are such that the scaled rectangle is centered at the origin.

Consider the tangent vector t at a point p on an edge curve in $\varGamma$ on the surface of $\mathbb{S}_1$ and the longitudinal vector l starting at p and pointing to the north pole. As the edge curve is strictly monotonous in y-direction $\langle t, l\rangle > 0$. The same holds for the corresponding vectors $t' = (t'_x, t'_y)$ and l' in $M[\varGamma]$ since M preserves angles. Let $t'' = (t''_x, t''_y)$ and l'' be the corresponding vectors in $(f \circ M)[\varGamma]$. Note that M maps longitudinals to vertical lines. Since, up to the translation, f is a combination of scalings in x- and y-direction, we have that $l'' = (0, 1)$ after a normalization. Although f is not angle-preserving, it does not change the sign of the corresponding scalar product in $(f \circ M)[\varGamma]$ since $\langle t'', l''\rangle = t''_x \cdot 0 + t''_y \cdot 1 = \frac{2}{y'_{\max} - y'_{\min}} t'_y = \frac{2}{y'_{\max} - y'_{\min}} \langle t', l'\rangle > 0$. Hence, the resulting edge curves respect H and we have an upward drawing of G on $(\mathbb{C}_s, H)$.

If a vertex v_{N} is placed at the north pole, then define $y_{\max}$ to be the maximum y-coordinate of any vertex in $V \setminus v_{\mathrm{N}}$ and define $\mathbb{S}'_1$ as above. The mapping $(f \circ M)$ is applied to $\varGamma \cap \mathbb{S}'_1$ to obtain $\varGamma'$. Note that $\varGamma'$ does not contain v_{N}. In $\varGamma'$ the edges to v_{N} are cut at the upper side of the fundamental polygon. We additionally shrink $\varGamma'$ in y-direction by $g : (x, y) \mapsto (x, \frac{1}{2}y)$. Note that in $g[\varGamma']$ all edges still respect H. In $g[\varGamma']$ we have obtained free space $B_{\mathrm{N}} = [-1, 1[\times]\frac{1}{2}, 1[$ in $\mathbb{C}_s$ with no points of $g[\varGamma']$. We place v_{N} somewhere in B_{N}, e.g., at $(0, \frac{3}{4})$, and reconnect all its incident edges by straight lines, which respect the homogeneous field. A similar procedure is applied when a vertex is placed at the south pole. For the converse direction, i. e., *(i)* $\Rightarrow$ *(ii)*, the proof is analogous by using the inverse of the transformation $(f \circ M)$.

For *(i)* $\Rightarrow$ *(iii)*, let $\varGamma$ be a drawing of $G \in [\![\mathbb{C}_s H]\!]_{\mathrm{s}}$. Intuitively, we bend the fundamental polygon containing $\varGamma$ such that the identified left and right sides actually mend. More formally, apply the map $f :]-1, 1[^2 \to \mathbb{R}^3 : (x, y) \mapsto (\cos x, y, \sin x)$ to $\varGamma$. As the y-coordinate is mapped onto itself and $\varGamma$ respects H pointing from bottom to top, all edges in $f[\varGamma]$ increase monotonically in the y-direction of the cylinder axis. The case *(iii)* $\Rightarrow$ *(i)* follows analogously, as essentially the inverse of f can be used.

For *(i)* $\Rightarrow$ *(iv)* consider the map

$$f : \mathbb{C}_s \to \mathbb{P} : (x, y) \mapsto \frac{y + 2}{4} \cdot (\cos(\pi x), \sin(\pi x)) . \tag{3}$$

Intuitively, f transforms the lateral surface of the rolling cylinder to a ring in the plane centered around the origin with inner radius $\frac{1}{4}$ and outer radius $\frac{3}{4}$. The bottom of the fundamental polygon $\mathbb{C}_s$ maps to the inner circular boundary and the top to the outer circular boundary of the ring. f is a conformal map and H is f-related to R, i. e., f preserves angles and maps H to R (Property 1) . By Proposition 1 we can conclude that any graph in $[\![\mathbb{C}_s H]\!]_{\mathrm{s}}$ is also in $[\![\mathbb{P} R]\!]_{\mathrm{s}}$.

For *(iv)* $\Rightarrow$ *(i)*, the inverse f^{-1} of f can be used. However, some care has to be taken if a vertex is placed at the origin $(0, 0)$ of $\mathbb{P}$. Then the same technique as with the sphere applies here as well.

□

Property 1. The smooth homeomorphism f defined in (3) is conformal and H and R are f-related up to normalization.

Proof. Function f given by

$$f : \mathbb{C}_s \to \mathbb{P} : (x, y) \mapsto \frac{y+2}{4} \cdot (\cos(\pi x), \sin(\pi x)) \,. \tag{4}$$

Function f is conformal [4]. To show that H and R are f-related up to normalization, we have to check that f transforms the homogeneous field to the radial field, i. e.,

$$J_f f^{-1}(x, y) H(f^{-1}(x, y)) \simeq R(x, y) \,, \tag{5}$$

J_f is the Jacobian matrix of $f(x, y) = (f_x(x, y), f_y(x, y))$ is defined as

$$J_f(x, y) = \begin{bmatrix} \frac{\partial f_x}{\partial x}(x, y) & \frac{\partial f_x}{\partial y}(x, y) \\ \frac{\partial f_y}{\partial x}(x, y) & \frac{\partial f_y}{\partial y}(x, y) \end{bmatrix}$$

and $H(x, y) = (0, 1)$ for all $x, y \in \mathbb{C}_s$. Hence, we get:

$$\begin{aligned} J_f f^{-1}(x, y) H(f^{-1}(x, y)) &= \begin{bmatrix} \frac{\partial f_x}{\partial x} f^{-1}(x, y) & \frac{\partial f_x}{\partial y} f^{-1}(x, y) \\ \frac{\partial f_y}{\partial x} f^{-1}(x, y) & \frac{\partial f_y}{\partial y} f^{-1}(x, y) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f_y}{\partial x} f^{-1}(x, y) \\ \frac{\partial f_y}{\partial y} f^{-1}(x, y) \end{bmatrix} . \end{aligned} \tag{6}$$

The inverse $f^{-1}(x, y) = (f_x^{-1}(x, y), f_y^{-1}(x, y))$ of f is given by:

$$\begin{aligned} f_x^{-1}(x, y) &= \frac{\operatorname{sgn} y}{\pi} \arccos \frac{x}{\sqrt{x^2 + y^2}} \,, \\ f_y^{-1}(x, y) &= 4\sqrt{x^2 + y^2} - 2 \,, \end{aligned} \tag{7}$$

and the two derivatives of $\partial f_x / \partial y$ and $\partial f_y / \partial y$ are given by:

$$\begin{aligned} \frac{\partial f_x}{\partial y} &= -\frac{\pi}{4} \cos \pi x \,, \\ \frac{\partial f_y}{\partial y} &= \frac{\pi}{4} \cos \pi x \,. \end{aligned} \tag{8}$$

Inserting Equations (7) and (8) into (6) yields:

$$J_f f^{-1}(x,y)H(f^{-1}(x,y)) = \begin{bmatrix} \frac{1}{4}\cos\pi\frac{\operatorname{sgn} y}{\pi}\arccos\frac{x}{\sqrt{x^2+y^2}} \\ \frac{1}{4}\sin\pi\frac{\operatorname{sgn} y}{\pi}\arccos\frac{x}{\sqrt{x^2+y^2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\cos\operatorname{sgn} y\arccos\frac{x}{\sqrt{x^2+y^2}} \\ \frac{1}{4}\sin\operatorname{sgn} y\arccos\frac{x}{\sqrt{x^2+y^2}} \end{bmatrix}.$$

By using the identities $\sin -\Theta = -\sin\Theta$, $\cos -\Theta = \cos\Theta$, $\cos\arccos\Theta = \Theta$, and $\sin\arccos\Theta = \sqrt{1-\Theta^2}$ we can further simplify this equation to:

$$J_f f^{-1}(x,y)H(f^{-1}(x,y)) = \begin{bmatrix} \frac{1}{4}\cos\arccos\frac{x}{\sqrt{x^2+y^2}} \\ \frac{\operatorname{sgn} y}{4}\sin\arccos\frac{x}{\sqrt{x^2+y^2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{4\sqrt{x^2+y^2}} \\ \frac{\operatorname{sgn} y}{4}\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2} \end{bmatrix} = \begin{bmatrix} \frac{x}{4\sqrt{x^2+y^2}} \\ \frac{\operatorname{sgn} y}{4}\frac{|y|}{\sqrt{x^2+y^2}} \end{bmatrix} = \frac{1}{4\sqrt{x^2+y^2}}\begin{bmatrix} x \\ y \end{bmatrix} \simeq \begin{bmatrix} x \\ y \end{bmatrix}$$

where the last vector is the radial field $R(x,y) = (x,y)$. □

Theorem 2. *A graph G is embeddable in the plane respecting the cyclic field if and only if G is embeddable on the rolling cylinder with the homogeneous field, i. e.,* $[\![\mathbb{P}C]\!]_s = [\![\mathbb{C}_r H]\!]_s$ *and* $[\![\mathbb{P}C]\!]_w = [\![\mathbb{C}_s H]\!]_w$.

Proof. The proof is analogous to the case *(i)* ⇔ *(iv)* in the proof of Theorem 1 except that for the functions f and g the coordinates x and y are swapped. □

Hashemi et al. have shown that deciding if a graph has an upward drawing on the sphere is **NP**-complete [13]. Limaye et al. [16] stated this problem as open on the cylinder. Theorem 1 solves this problem.

Corollary 1. *Upward planarity testing on the cylinder is* **NP***-hard.*

Longitudinal cycles are permitted in **RUP**, whereas **SUP** contains only acyclic graphs. Thus, **RUP** is stronger than **SUP**. Even more, this is also true if we consider only acyclic graphs.

Theorem 3. ***SUP*** $\subsetneq$ ***RUP****, even for acyclic graphs.*

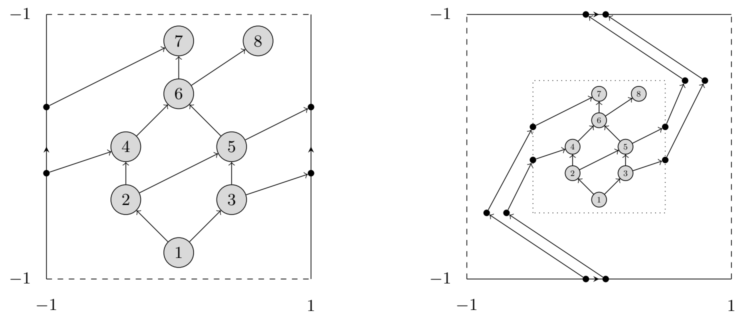
Proof. Consider a graph $G \in \mathbf{SUP}$ along with its drawing Γ on $\mathbb{C}_s$ with the homogeneous field. Then G is acyclic. To show that $G \in \mathbf{RUP}$ we give a step-by-step transformation of Γ to a drawing on $\mathbb{C}_r$ which respects the homogeneous field H.

First we straighten Γ into a polyline drawing, which is then transformed from the standing onto the rolling cylinder while upward planarity is preserved. Cut Γ at the y-coordinates of the vertices. Each cut defines a ring of points, which are the x-coordinates of the vertices, and temporarily introduce a dummy vertex for each crossing of an edge with the cut. A *slice* consists of the region of Γ between two adjacent cuts. It has a lower and an upper ring of (dummy) vertices and a planar upward routing of segments of edges between the rings. We process slices iteratively from bottom to top. For a slice S take an edge segment connecting two (dummy) vertices, say p_1 on the lower ring and q_1 on the upper ring. Now rotate the upper ring such that p_1 and q_1 have the same x-coordinate. Replace each edge segment from a (dummy) vertex p on the lower ring to a (dummy) vertex on the upper ring by a straight line, such that the cyclic order of the incident edges of each vertex is preserved. Since two curves did not cross before, they cannot cross after the straightening, because the relative order of their endpoints on the rings with respect to (p_1, q_1) is preserved. (One can make (p_1, q_1) the boundary of the fundamental polygon.)

Now let Γ be the so obtained polyline drawing. In the remainder of the proof we need that all edges that cross the vertical line $x = -1$ leave the fundamental polygon to the right and enter it from the left, i. e., the x-value of the edge curves immediately before their crossing is positive and negative immediately afterwards. According to Lemma 5 of [2] by identifying all edges with *inner segments* a polyline drawing on $\mathbb{C}_s$ can always be transformed such that this condition holds, which we assume to hold for Γ as well.

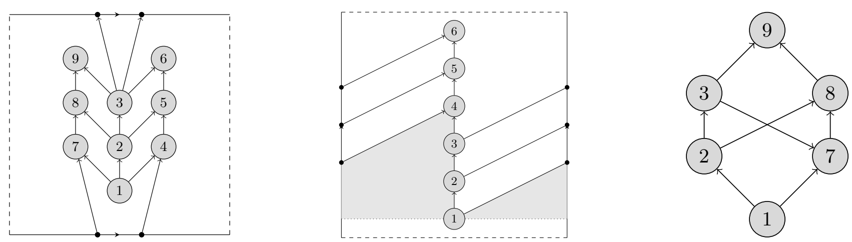
Let $f : \mathbb{C}_s \to \mathbb{C}_r : (x, y) \mapsto \frac{1}{2}(x, y)$ be the scaling which shrinks by $\frac{1}{2}$ and consider the drawing $f[\Gamma]$ on $\mathbb{C}_r$. Since the scalar product is linear and the scaling factor $\frac{1}{2} > 0$, $f[\Gamma]$ still respects the homogeneous field H. For instance, the drawing of Fig. 1(a) is scaled to the drawing in the dotted rectangle in Fig. 1(b). It remains to show how to reconnect the formerly identical points on the left and right boundary of $f[\Gamma]$ by field-respecting edges in $\mathbb{C}_r$. Let $y_1 < y_2 < \ldots < y_k$ be the ascending y-coordinates of the points $r_i = (\frac{1}{2}, y_i)$ and $l_i = (-\frac{1}{2}, y_i)$ on the right and left boundary in $f[\Gamma]$, respectively. Define points $r'_i = (\frac{3}{4} - \frac{y_i}{4}, \frac{1}{2})$ and $l'_i = (-\frac{3}{4} - \frac{y_i}{4}, -\frac{1}{2})$ with $1 \leq i \leq k$. Connect r_i to r'_i by a straight-line segment. Note that these segments do not intersect since $y_i < y_j \Leftrightarrow x(r'_i) > x(r'_j)$ for $i \neq j$. Analogously, connect all l'_i to l_i by non-intersecting segments. As $-\frac{1}{2} < y_i < \frac{1}{2}$, all (directed) line-segments strictly follow H. Finally, connect all r'_i to l'_i. These line-segments also strictly follow H and are non-intersecting since $x(r'_i) < x(r'_j) \Leftrightarrow x(l'_i) < x(l'_j)$. The result of the whole process applied to Fig. 1(a) is depicted in Fig. 1(b). □

Proposition 2 ( ). *On the rolling cylinder with the homogeneous field, the class of (strictly) upward embeddable graphs coincides with the class of weakly upward embeddable graphs, i. e.,* $[\![\mathbb{C}_r H]\!]_s = [\![\mathbb{C}_r H]\!]_w$.



(a) **SUP**-drawing (b) **RUP**-drawing

Fig. 1. Transformation from the standing to the rolling cylinder



(a) G on $(\mathbb{C}_r, H)$ (b) Subgraph G' on $(\mathbb{C}_s, H)$ (c) Subgraph G''

Fig. 2. An acyclic graph $G \in$ **RUP** but not in **SUP**

Proof. The number of windings of an edge can be bounded from above by n, since each edge can be restricted such that it winds at most n times around the rolling cylinder. Consider a latitudinal line that does not cross any vertex. Split each edge by dummy vertices at the points the edge crosses this line. An *inner segment* connects two dummy vertices. Consider two vertices v_l and v_r of G with w. l. o. g. $x(v_l) < x(v_r)$ such that there are at least two inner segments of an edge between them. Let s_l be the inner segment immediately to the right of v_l and s_r the inner segment immediately to the left of v_r. Choose v_l and v_r such that no other vertex of G lies between s_l and s_r. The inner segments form a *bundle* $B_l = (s_l = s_1, \ldots, s_k)$ such that all s_i $(1 \geq i \geq k)$ belong to distinct edges of G, s_{i+1} is the immediate right successor of s_i $(1 \leq i < k)$, and k is maximal. Let e_k be the edge of s_k. Let s'_k be the last segment of e_k to the left of v_r and let $B_r = (s'_1, \ldots, s'_k = s_r)$. Then the edges associated with the segments of the bundle B_l wind around the cylinder and the winding begins with B_l and ends at B_r. Intuitively, consider B_l and shift it by k to the right. Then, due to planarity, in the shifted bundle the associated edges are the same as in B_l. The same is true for B_r when shifted to the left. This also holds when shifting indefinitely. Thus, all bundles preserve the order of the edges. Now replace all inner segments from B_l to B_r including B_l and B_r by a new bundle $B = (s''_1, \ldots, s''_k)$ from the start dummy vertices of the inner segments of B_l to the end vertices of the inner segments of B_r. The new inner segments can always be routed (not necessarily straight-line) in the region of the original bundles such that they do not cross each other or any other point in drawing. We execute this procedure for each of the at most $n-1$ vertex pairs v_l and v_r. After each step, at least one additional edge has the reduced number of windings equal to 1.

For the second step let Γ be a drawing of G on $\mathbb{C}_r$ respecting H after the previously described procedure. We consider Γ as a level graph with latitudinal levels through the vertices of G. In the weak case segments of some edges (u, v) are parallel to the latitudes. They will be slanted slightly, and they are straight lines. The segments of edge curves between adjacent levels will be replaced by straight-line segments on the fundamental polygon.

The *points* on a level l of Γ are all vertices on l, the points where an edge enters or leaves l or properly crosses l. These points are ordered left-to-right by the values of their x-coordinates. Let s be the maximal number of points on any level which is bounded by the bound on the number of windings.

On each level consider the directed subgraph G_l. The vertices of G_l are the points on l and the edges are the latitudinal segments. G_l is acyclic. Sort it topologically (from zero) and assign each point p its topsort level t_p.

Let δ_h be the least horizontal distance between any two points on any level and let δ_v be the least vertical distance between levels. Then any straight line between any two adjacent levels has a slope of at most $\frac{\delta_h}{2}$. Choose Δ such that $\Delta < \frac{\delta_h \delta_v}{2} \max_p\{t_p\}$. Lift each point p on l by $t_p \Delta$, and connect any two points by a straight line if they were connected before by a segment of an edge curve. Then all these lines are upward with respect to H, and any two such lines cannot cross. By the planarity of Γ the endpoints of two such lines on are properly ordered on their levels, and Δ is chosen adjacent lines cannot cross.

Observe that the so constructed drawing of G is a polyline drawing. □

Forthcoming we shall establish proper inclusions among the main classes of upward drawable graphs. For the plane and the sphere this has been proved at several places and it comes from the distinction by the *st*-edge. The graph in Fig. 2(c) serves as a counterexample.

The *2-wing graph* displayed in Fig. 2(a) is an acyclic **RUP** graph which is not planar upward drawable on the sphere or the standing cylinder. It is 3-connected and due to the upward drawing its embedding is unique. Let G' be the subgraph of G induced by the vertices $\{1, 2, 3, 4, 5, 6\}$ which are connected by the path $P = (1, 2, 3, 4, 5, 6)$. On $(\mathbb{C}_s, H)$ the vertices along P must be placed with strictly increasing y-coordinate due to H. In Fig. 2(b) G' is drawn on $(\mathbb{C}_s, H)$ using the same embedding as in Fig. 2(a). The remaining vertices $\{7, 8, 9\}$ of G must all be placed above vertex 1, since there is path from 1 to 7, 8, and 9. Due to the uniqueness of the embedding, the vertices $\{7, 8, 9\}$ must be placed within the shaded area in Fig. 2(b). This area is homeomorphic to the plane $\mathbb{P}$. Hence, if $\{7, 8, 9\}$ could be placed within the shaded area without crossings, then the subgraph G'' of G induced by the vertices $\{1, 2, 3, 7, 8, 9\}$ would have an

(a) G on $(\mathbb{P}, N)$ (b) G on $(\mathbb{C}_r, H)$

Fig. 3. A planar graph G with $G \notin$ **RUP**

embedding on $\mathbb{P}$ respecting H, i. e., $G'' \in$ **UP**. However, G'' is isomorphic to the graph displayed in Fig. 2(c) which is known not to be in **UP** [5].

In **wSUP** latitudinal cycles are allowed and therefore **wSUP** properly contains **UP** and **SUP** as the latter two only allow acyclic graphs. Also **RUP** allows cycles, which implies similar proper inclusions.

The vertices of two cycles with one common vertex must have the same y-coordinate on $\mathbb{C}_s$ with H. In contrast, this graph can easily be embedded on $\mathbb{C}_r$ with H. Thus, **SUP** $\subsetneq$ **RUP**. Further, K_5 can be embedded on the torus and, hence, **P** $\subsetneq$ **T**. Finally, the *wheel graph* as shown in Fig. 3(a) shows that upward planarity on a rolling cylinder is a proper restriction over planarity. As special techniques apply, this is stated as our next lemma.

Lemma 1. ***RUP*** $\subsetneq$ ***P***

Proof. **RUP** $\subseteq$ **P** since the rolling cylinder is a surface of genus 0. For the proper inclusion consider the planar graph G depicted in Fig. 3(a). We show that $G \notin$ **RUP**. G has a Hamiltonian cycle $\mathcal{C} = (1, 2, 3, 4, 5, 1)$. Note that any cycle embedded on $\mathbb{C}_r$ with the homogeneous field wraps exactly once around the cylinder, i. e., its winding number is 1. Its winding number is greater 0 since otherwise its start and endpoint could not connect and it must be less than 2 since otherwise the edge curve would be self-intersecting. As all other edges in Fig. 3(a) follow the direction of $\mathcal{C}$ and start and end at distinct vertices of $\mathcal{C}$, their winding number on $\mathbb{C}_r$ is 0. Consider the embedding of G on $\mathbb{C}_r$ displayed in Fig. 3(b), where edge $(3, 1)$ is drawn dotted. $\mathcal{C}$ divides C_r into a left- and a right-hand region. To avoid a crossing between the edges $(1, 4)$ and $(2, 5)$, they must lie in different regions, e. g., $(1, 4)$ to the right and $(2, 5)$ to the left of $\mathcal{C}$. Now consider the region R enclosed by the edges $(1, 2), (2, 5), (4, 5), (1, 4)$, which contains vertex 3. The curve of edge $(3, 1)$ must start within R and, due to the homogeneous field, must reach vertex 1 from below. Thus, the curve of edge $(3, 1)$ starts within R and ends outside of R, which always causes a crossing. $\square$

Theorem 4. *Let* $\boldsymbol{DAG}$ *be the set of all acyclic graphs. The classes of graphs are related as follows.*

$$\begin{array}{ccccccccc}
\boldsymbol{UP} & \subsetneq & \boldsymbol{SUP} & \subsetneq & \boldsymbol{RUP} \cap \boldsymbol{DAG} & \subsetneq & \boldsymbol{RUP} & \subsetneq & \boldsymbol{UT} \\
\shortparallel & & \rotatebox{90}{$\subsetneq$} & & & & \shortparallel & & \rotatebox{90}{$\subseteq$} \\
\boldsymbol{wUP} & & \boldsymbol{wSUP} & & & & \boldsymbol{wRUP} & & \boldsymbol{wUT} \\
 & & & & & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} \\
 & & & & & & \boldsymbol{P} & \subsetneq & \boldsymbol{T}
\end{array} \qquad (9)$$

Finally, we classify the work of Dolati et al. [8] on upward drawings on the lying and on the standing torus, where in each case the edges respect the south-north direction. On the lying torus the south (north) pole is a ring consisting of all y-minimal (y-maximal) points of the torus. This corresponds to our notion of the antiparallel field (see Tab. 1) and the graph class $[\![\mathbb{T}A]\!]_{\mathrm{s}}$. On the standing cylinder the south (north) pole is the single point with minimal (maximal) y-coordinate. In our classification this is the radial field and the graph class $[\![\mathbb{T}R]\!]_{\mathrm{s}}$. The authors showed that $[\![\mathbb{T}A]\!]_{\mathrm{s}} \subsetneq [\![\mathbb{T}R]\!]_{\mathrm{s}}$ and state that the time complexity of deciding whether or not a graph is in (one of) the two sets is unknown.

4 Complexity

Finally we address the recognition problems for upward drawability, which are known to be **NP**-hard for the plane and sphere and, hence, the standing cylinder. It is also **NP**-hard for the torus, and still remains open for the rolling cylinder.

Theorem 5. *Deciding whether or not a graph* $G \in \boldsymbol{UT}$ *is* **NP***-complete, even if* G *is connected.*

Proof. If the graph does not have to be connected, simply reduce from **UP** by adding to G a suitably directed K_7. Any embedding of the K_7 must be two-cell, so all remaining faces have genus 0. Thus $G \cup K_7 \in \mathbf{UT} \Leftrightarrow G \in \mathbf{UP}$. For connected graphs reconstruct the **NP**-completeness proof of **UP**. The constructed graph candidate for **UP** has a dedicated vertex v lying on the outside of the graph. Add an edge e from any of the K_7 vertices to v. Again, $G \cup K_7 \in \mathbf{UT} \cup \{e\} \in \mathbf{UT} \Leftrightarrow G \in \mathbf{UP}$. □

5 Future Work

We have classified upward planarity on the plane, sphere, standing and rolling cylinder and the torus. We wish to characterize the new classes **wSUP**, **RUP**, **wRUP**, **UT**, and **wUT** in a similar way as it is known for the plane and the sphere, and we wish to establish efficient recognition algorithms for restrictions with fixed embeddings or single sources. This is under work.

References

1. C. Auer, C. Bachmaier, F. J. Brandenburg, W. Brunner, and G. Andreas. Plane drawings of queue and deque graphs. In U. Brandes and S. Cornelsen, editors, *GD 2010*, volume 6502 of *LNCS*, pages 68–79, 2011.
2. C. Bachmaier. A radial adaption of the sugiyama framework for visualizing hierarchical information. *IEEE Trans. Vis. Comput. Graphics*, 13(3):583–594, 2007.
3. C. Bachmaier, F. J. Brandenburg, W. Brunner, and R. Fülöp. Coordinate assignment for cyclic level graphs. In H. Q. Ngo, editor, *COCOON 2009*, volume 5609 of *LNCS*, pages 66–75, 2009.
4. H. Cohn. *Conformal Mapping on Riemann Surfaces.* Dover, 1967.
5. G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing: Algorithms for the Visualization of Graphs.* Prentice Hall, 1999.
6. A. Dolati. Digraph embedding on t_h. In *Cologne-Twente Workshop on Graphs and Combinatorial Optimization, CTW 2008*, pages 11–14, 2008.
7. A. Dolati and S. M. Hashemi. On the sphericity testing of single source digraphs. *Discrete Math.*, 308(11):2175–2181, 2008.
8. A. Dolati, S. M. Hashemi, and M. Kosravani. On the upward embedding on the torus. *Rocky Mt. J. Math.*, 38(1):107–121, 2008.
9. S. Foldes, I. Rival, and J. Urrutia. Light sources, obstructions and spherical orders. *Discrete Math.*, 102(1):13–23, 1992.
10. A. Gard and R. Tamassia. On the computational complexity of upward and rectilinear planarity testing. *SIAM Journal on Computing*, 31(2):601–625, 2001.
11. K. A. Hansen. Constant width planar computation characterizes ACC^0. *Theor. Comput. Sci.*, 39(1):79–92, 2006.
12. S. M. Hashemi. Digraph embedding. *Discrete Math.*, 233(1–3):321–328, 2001.
13. S. M. Hashemi, I. Rival, and A. Kisielewicz. The complexity of upward drawings on spheres. *Order*, 14:327–363, 1998.
14. J. M. Lee. *Introduction to Smooth Manifolds.* Springer, 2002.
15. N. Limaye, M. Mahajan, and J. M. N. Sarma. Evaluating monotone circuits on cylinders, planes and tori. In B. Durand and W. Thomas, editors, *STACS 2006*, volume 3884 of *LNCS*, pages 660–671. Springer, 2006.
16. N. Limaye, M. Mahajan, and J. M. N. Sarma. Upper bounds for monotone planar circuit value and variants. *Comput. Complex.*, 18(3):377–412, 2009.
17. J. E. Marsen, T. Ratiu, and R. Abraham. *Manifolds, Tensor Analysis, and Applications.* Springer, 3rd edition, 2001.
18. W. S. Massey. *Algebraic Topology: An Introduction.* Springer, 1967.
19. B. Mohar and P. Rosenstiel. Tessellation and visibility representations of maps on the torus. *Discrete Comput. Geom.*, 19:249–263, 1998.
20. B. Mohar and C. Thomassen. *Graphs on Surfaces.* John Hopkins University Press, 2001.
21. J. P. Snyder. Map projections – a working manual. *US Geological Survey*, 1395, 1987.
22. K. Sugiyama, S. Tagawa, and M. Toda. Methods for visual understanding of hierarchical system structures. *IEEE Trans. Syst., Man, Cybern.*, 11(2):109–125, 1981.
23. C. Thomassen. Planar acyclic oriented graphs. *Order*, 5(1):349–361, 1989.
24. I. Wegener. *Complexity Theory - Exploring the Limits of Efficient Algorithms.* Springer, 2005.