NIC-Planar Graphs[☆]

Christian Bachmaier[a], Franz J. Brandenburg[a], Kathrin Hanauer[a,*], Daniel Neuwirth[a], Josef Reislhuber[a]

[a] *University of Passau*
94030 Passau, Germany

Abstract

A graph is NIC-planar if it admits a drawing in the plane with at most one crossing per edge and such that two pairs of crossing edges share at most one common end vertex. NIC-planarity generalizes IC-planarity, which allows a vertex to be incident to at most one crossing edge, and specializes 1-planarity, which only requires at most one crossing per edge.

We characterize embeddings of maximal NIC-planar graphs in terms of generalized planar dual graphs. The characterization is used to derive tight bounds on the density of maximal NIC-planar graphs which ranges between $3.2(n-2)$ and $3.6(n-2)$. Further, we prove that optimal NIC-planar graphs with $3.6(n-2)$ edges have a unique embedding and can be recognized in linear time, whereas the general recognition problem of NIC-planar graphs is $\mathcal{NP}$-complete. In addition, we show that there are NIC-planar graphs that do not admit right angle crossing drawings, which distinguishes NIC-planar from IC-planar graphs.

Keywords: NIC-planarity, density, recognition, 1-planarity, right-angle crossings (RAC)

1. Introduction

Beyond-planar graphs, a family of graph classes defined as extensions of planar graphs with different restrictions on crossings, have received recent interest [32]. 1-planar graphs constitute an important class of this family. A graph is *1-planar* if it can be drawn in the plane with at most one crossing per edge. These graphs were introduced by Ringel [34] in the context of coloring a planar graph and its dual simultaneously and have been studied intensively since then. Ringel observed that a pair of crossing edges can be augmented by planar

[☆]Supported by the Deutsche Forschungsgemeinschaft (DFG), grant Br835/18-1.

[*]Corresponding author.
Email addresses: bachmaier@fim.uni-passau.de (Christian Bachmaier), brandenb@informatik.uni-passau.de (Franz J. Brandenburg), hanauer@fim.uni-passau.de (Kathrin Hanauer)

Table 1: The density of maximal graphs. An asterisk marks our results.

	1-planar	NIC-planar	IC-planar	outer 1-planar
upper bound	$4n-8$ [10, 11]	$\frac{18}{5}(n-2)$ [38], (*)	$\frac{13}{4}n-6$ [30]	$\frac{5}{2}n-2$ [5]
$\lfloor$ example	$4n-8$ [10, 11]	$\frac{18}{5}(n-2)$ [21], (*)	$\frac{13}{4}n-6$ [39]	$\frac{5}{2}n-2$ [5]
lower bound	$\frac{20}{9}n-\frac{10}{3}$ [9]	$\frac{16}{5}(n-2)$ (*)	$3n-5$ [7]	$\frac{11}{5}n-\frac{18}{5}$ [5]
$\lfloor$ example	$\frac{45}{17}n-\frac{84}{17}$ [16]	$\frac{16}{5}(n-2)$ (*)	$3n-5$ [7]	$\frac{11}{5}n-\frac{18}{5}$ [5]

edges to form K_4. Bodendiek et al. [10, 11] proved that 1-planar graphs with n vertices have at most $4n-8$ edges, which is a tight bound for $n=8$ and all $n \geq 10$. These facts have also been discovered in other works. 1-planar graphs with $4n-8$ edges are called optimal [11, 35]. They have a special structure and consist of a triconnected planar quadrangulation with a pair of crossing edges in each face. A graph G is maximal 1-planar if no further edge can be added to G without violating 1-planarity. Brandenburg et al. [16] found sparse maximal 1-planar graphs with less than $2.65n$ edges, which implies that there are maximal 1-planar graphs that are not optimal and that are even sparser than maximal planar graphs. The best known lower bound on the density of maximal 1-planar graphs is $2.22n$ [9] and neither the upper nor the lower bound are known to be tight.

There are some important subclasses of 1-planar graphs. A graph is *IC-planar* (independent crossing planar) [3, 15, 30, 39] if it has a 1-planar embedding so that each vertex is incident to at most one crossing edge. IC-planar graphs were introduced by Albertson [3] who studied the coloring problem. Král and Stacho [30] solved the coloring problem and proved that K_5 is the largest complete graph that is IC-planar. IC-planar graphs have an upper bound of $3.25n-6$ on the number of edges, which is known as a tight bound as there are optimal IC-planar graphs with $13k-6$ edges for all $n=4k$ and $k \geq 2$ [39]. For other values of n with $n \geq 8$ the maximum number of edges is $\lfloor 3.25n-6 \rfloor$. On the other hand, there are sparse maximal IC-planar graphs with only $3n-5$ edges for all $n \geq 5$ [7]. In *NIC-planar* graphs (near-independent crossing planar) [38], two pairs of crossing edges share at most one vertex. Equivalently, if every pair of crossing edges is augmented by planar edges to K_4, an edge may be part of at most one K_4 that is embedded with a crossing. Note that a graph is NIC-planar if every biconnected component is NIC-planar. NIC-planar graphs were introduced by Zhang [38], who proved a density of at most $3.6(n-2)$ and showed that K_6 is not NIC-planar. Czap and Šugarek [21] give an example of an optimal NIC-planar graph with 27 vertices and 90 edges which proves that the upper bound is tight. *Outer 1-planar* graphs are another subclass of 1-planar graphs. They must admit a 1-planar embedding such that all vertices are in the outer face [5, 28]. Results on the density of maximal graphs are summarized in Table 1.

There is a notable interrelationship between 1-planar and RAC graphs which are graphs that can be drawn straight-line with right angle crossings [23, 26].

Didimo et al. [23] showed that RAC graphs have at most $4n - 10$ edges and proved that there are *optimal* RAC graphs with $4n - 10$ edges for all $n = 3k + 5$ and $k \geq 3$. For other values of n it is unknown whether there are optimal RAC graphs. Eades and Liotta [26] established that optimal RAC graphs (they called them maximally dense) admit a special structure and proved that optimal RAC graphs are 1-planar. However, not all RAC-graphs are 1-planar and vice versa [26], i. e., the classes of RAC-graphs and 1-planar graphs are incomparable. Recently, Brandenburg et al. [15] showed that every IC-planar graph admits a RAC drawing, which implies that every IC-planar graph is a RAC graph. They posed the problem whether NIC-planar graphs are RAC graphs, which we refute. Hence, with respect to RAC drawings, NIC-planar graphs behave like 1-planar graphs and differ from IC-planar graphs.

Recognizing 1-planar graphs is $\mathcal{NP}$-complete in general [27, 29], and remains $\mathcal{NP}$-complete even for graphs of bounded bandwidth, pathwidth, or treewidth [8], if an edge is added to a planar graph [17], and if the input graph is triconnected and given with a rotation system [6]. Likewise, testing IC-planarity is $\mathcal{NP}$-complete [15]. On the other hand, there are polynomial-time recognition algorithms for 1-planar graphs that are maximized in some sense, such as triangulated [19], maximal [12], and optimal graphs [14].

In this paper we study NIC-planar graphs. After some basic definitions in Sect. 2, we characterize embeddings of maximal NIC-planar graphs in terms of generalized planar dual graphs in Sect. 3, and derive tight upper and lower bounds on the density of maximal 1-planar graphs in Sect. 4 for infinitely many values of n. A linear-time recognition algorithm for optimal NIC-planar graphs is established in Sect. 5. In Sect. 6 we show that NIC-planar graphs are incomparable with RAC graphs, and we consider the recognition problem for NIC-planar graphs. We conclude in Sect. 8 with some open problems.

2. Preliminaries

We consider simple, undirected graphs $G = (V, E)$ with n vertices and m edges and assume that the graphs are biconnected. The subgraph induced by a subset $U \subseteq V$ of vertices is denoted by $G[U]$.

A *drawing* $\mathcal{D}(G)$ is a mapping of G into the plane such that the vertices are mapped to distinct points and each edge is mapped to a Jordan arc connecting the points of the end vertices. Two edges *cross* if their Jordan arcs intersect in a point different from their end points. Crossings subdivide an edge into two or more uncrossed *edge segments*. An edge without crossings is called *planar* and consists only of a *trivial* edge segment. Edge segments of crossed edges are said to be *non-trivial*. For a crossed edge $\{u, v\}$, we denote the extremal edge segment that is incident to u (v) by $\{\underline{u}, v\}$ ($\{u, \underline{v}\}$). A drawing is *planar* if every edge is planar, and *1-planar* if there is at most one crossing per edge.

A drawing of a graph partitions the plane into empty regions called *faces*. A face is defined by the cyclic sequence of edge segments that forms its *boundary*, which is described by vertices and crossing points, e. g., face f_{ab} in Fig. 1a. Two faces f and g are said to be *adjacent*, denoted as $f \mid g$, if their boundaries share a common edge segment. A vertex v is *incident* to a face f if there is an edge $\{u, v\}$ such that f is either bounded by the trivial edge segment $\{u, v\}$ or the non-trivial edge segment $\{u, \underline{v}\}$. A face is called a (*trivial*) *triangle*, if its boundary consists of exactly three (trivial) edge segments. The set of all faces describes the *embedding* $\mathcal{E}(G)$. The embedding resulting from a planar (1-planar) drawing is called *planar* (*1-planar*). A 1-planar embedding coincides with the embedding of the *planarization* of G which is obtained by treating each crossing point as a vertex and the edge segments as edges. A 1-planar embedding is *triangulated* if every face is a triangle [19]. A *planar reduction* $\widehat{G}_{\mathcal{E}} \subseteq G$ is a planar subgraph of G obtained by removing exactly one edge of each pair of edges that cross in $\mathcal{E}(G)$. The *planar skeleton* $\widetilde{G}_{\mathcal{E}} \subseteq G$ is the planar subgraph of G with respect to $\mathcal{E}(G)$ that is obtained by removing all pairs of crossing edges.

We consider NIC-planar graphs and embeddings that are maximized in some sense. A NIC-planar embedding $\mathcal{E}(G)$ is *maximal (planar-maximal) NIC-planar* if no further (planar) edge can be added to $\mathcal{E}(G)$ without violating NIC-planarity or simplicity. A NIC-planar graph G is *maximal* if the graph $G + e$ obtained from G by the addition of an edge e is no longer NIC-planar. A graph is called a *sparsest* (*densest*) NIC-planar graph if it is maximal NIC-planar with the minimum (maximum) number of edges among all maximal NIC-planar graphs of the same size. A NIC-planar graph G is called *optimal* if G has exactly the upper bound of $3.6(n-2)$ edges. There are analogous definitions for other graph classes.

The concepts planar-maximal embedding and maximal and optimal graphs coincide for planar graphs, where the maximum number of edges is always $3n-6$. However, they are different for 1-planar, NIC-planar, and IC-planar graphs. The complete graph on five vertices without one edge, $K_5 - e$, is planar and has a planar embedding which is planar-maximal. Nevertheless, $K_5 - e$ can be embedded with a pair of crossing edges and e can be added as a planar edge. A graph is maximal if every embedding is maximal. All the same, an embedding $\mathcal{E}(G)$ of a graph G may be maximal 1-planar (NIC-planar, IC-planar) without G being maximal. As mentioned before, there are sparse 1-planar graphs with less than $2.65n$ edges, whereas optimal 1-planar graphs have $4n - 8$ edges [37]. Due to integrality, optimal NIC-planar graphs exist only for $n = 5k + 2$ and optimal IC-planar graphs only for $n = 4k$. Zhang and Liu [39] present optimal IC-planar graphs with $4k$ vertices and $13k - 6$ edges for every $k \geq 2$ vertices and Czap and Šugarek [21] gave an example of an optimal NIC-planar graph of size 27. We show that there are optimal NIC-planar graphs for all $n = 5k + 2$ with $k \geq 2$ and not for $n = 7$. The distinction between densest and optimal graphs is important, since optimal graphs often have a special structure, as we shall show for NIC-planar graphs.

The complete graph on four vertices K_4 plays a crucial role in 1-planar (IC- and NIC-planar) graphs. It has exactly the two 1-planar embeddings depicted in Fig. 1 [31]. If K_4 is a subgraph of another graph G, further vertices and edges of G may be inside the shown faces. Let $\mathcal{E}(G)$ be a NIC-planar embedding of

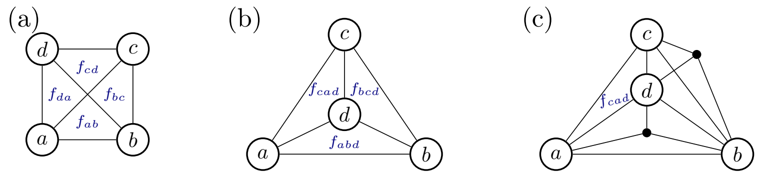

Figure 1: The two embeddings of a K_4 induced by the vertices $\{a, b, c, d\}$ (up to isomorphism): A kite (a) and a tetrahedron, which can be simple (b) or non-simple (c).

a graph $G = (V, E)$ and let $U = \{a, b, c, d\} \subseteq V$ such that $G[U]$ is K_4. Denote by $\mathcal{E}(G[U])$ the embedding of $G[U]$ induced by $\mathcal{E}(G)$. $G[U]$ is embedded as a *kite* in $\mathcal{E}(G)$ (see also Fig. 1a) if, w. l. o. g., $\{a, c\}$ and $\{b, d\}$ cross each other and there are faces $f_{ab}, f_{bc}, f_{cd}, f_{da}$ in $\mathcal{E}(G)$ such that f_{ab} is bounded exactly by $\{a, b\}$, $\{\underline{a}, c\}$, and $\{\underline{b}, d\}$, and analogously for f_{bc}, f_{cd}, f_{da}. Hence, there is no other vertex in the interior of a kite. $G[U]$ is embedded as a *tetrahedron* in $\mathcal{E}(G)$ if all edges are planar with respect to $\mathcal{E}(G[U])$ but not necessarily in $\mathcal{E}(G)$. The tetrahedron embedding of $G[U]$ in $\mathcal{E}(G)$ is called *simple*, if, w. l. o. g., d has vertex degree three and f_{abd}, f_{bcd}, and f_{cad} are faces in $\mathcal{E}(G)$. Then d is called the *center* of the tetrahedron. Fig. 1b shows a simple tetrahedron embedding of $G[U]$, whereas the tetrahedron embedding in Fig. 1c is non-simple due to the missing faces f_{abd} as well as f_{bcd}.

3. The Generalized Dual of Maximal NIC-planar Graphs

In this section, we study the structure of NIC-planar embeddings of maximal NIC-planar graphs. We use the results for tight upper and lower bounds of the density of NIC-planar graphs in Sect. 4 and for a linear-time recognition algorithm for optimal NIC-planar graphs in Sect. 5. The upper bound on the density was proved in [38] using a different technique and there is a construction for the tightness of the bound in [21] in steps of 15 vertices.

Let $\mathcal{E}(G)$ be a NIC-planar embedding of a maximal NIC-planar graph G. The first property we observe is that the subgraph of G induced by the end vertices of a pair of crossing edges induces K_4 with a kite embedding. It has been discovered by Ringel [34] and in many other works that a pair of crossing edges admits such a 1-planar embedding. This fact is used for a normal form of embeddings of triconnected 1-planar graphs [2] in the sense that it can always be established. With respect to maximal NIC-planarity, every embedding obeys to this normal form.

Lemma 1. *Let $\{a, c\}$, $\{b, d\}$ be two edges crossing each other in a NIC-planar embedding of a maximal NIC-planar graph $G = (V, E)$.*
Then, $\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\} \in E$ and the induced K_4 is embedded as a kite.

Proof. Let $\mathcal{E}(G)$ be a NIC-planar embedding of G. Consider a pair of vertices $e \in \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}$. As G is maximal, $e \in E$, otherwise it could be added without violating NIC-planarity. Thus $G[\{a, b, c, d\}]$ is K_4.

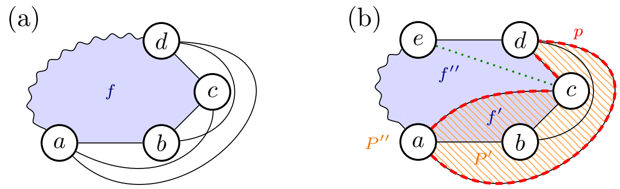

Figure 2: Proof of Lemma 2.

W. l. o. g., let $e = \{a, b\}$ (the other cases are similar). Due to the crossing of $\{a, c\}$ and $\{b, d\}$, the vertices a and b cannot be incident to another pair of crossing edges. Hence, e must be planar in $\mathcal{E}(G)$. Let f denote the face of $\mathcal{E}(G)$ that is incident to both a and b and bounded by the edge segments $\{\underline{a}, c\}$ and $\{\underline{b}, d\}$. Suppose that f is not bounded by $\{a, b\}$. Furthermore, e and the edge segments $\{\underline{a}, c\}$ and $\{\underline{b}, d\}$ form a closed path p that partitions the set of faces of $\mathcal{E}(G)$. Let g and h denote the faces bounded by $\{a, b\}$ in $\mathcal{E}(G)$. The fact that a and b cannot be incident to another pair of crossing edges implies that there is a vertex $x \neq a, b$ incident to g as well as a vertex $y \neq a, b$ incident to h. As p places g and h in different partitions and contains neither x nor y, $x \neq y$. Moreover, p consists of one planar edge and two edge segments and thus cannot be crossed. Subsequently, x and y are not adjacent.

Construct a new embedding $\mathcal{E}'(G)$ from $\mathcal{E}(G)$ by re-routing $\{a, b\}$ such that it subdivides f. Then, g and h conflate into one face gh in $\mathcal{E}'(G)$. As x and y are both incident to gh in $\mathcal{E}'(G)$, the edge $\{x, y\}$ can be added to G and embedded planarly such that it subdivides gh. The resulting embedding is NIC-planar, thus contradicting the maximality of G. □

Lemma 1 implies that every face bounded by at least one non-trivial edge segment is part of a kite and hence a non-trivial triangle. In fact, every embedding of a maximal NIC-planar graph is triangulated.

Lemma 2. *Every face of a NIC-planar embedding $\mathcal{E}(G)$ of a maximal NIC-planar graph G with $n \geq 5$ is a triangle.*

Proof. Let $\mathcal{E}(G)$ be a NIC-planar embedding of $G = (V, E)$ and let f be a face of $\mathcal{E}(G)$. Recall from Sect. 2 that a triangle is either trivial if its boundary consists only of trivial edge segments, i. e., planar edges, or non-trivial otherwise.

By Lemma 1, every face whose boundary contains non-trivial edge segments is part of a kite. Hence, if f is bounded by at least one non-trivial edge segment, it is a non-trivial triangle.

Otherwise, assume that f is bounded only by planar edges and suppose that f is not a triangle. Let $a, b, c, d \in V$ be distinct vertices incident to f such that f's boundary contains the edges $\{a, b\}, \{b, c\}, \{c, d\}$. Suppose that G does not contain edge $\{a, c\}$. Then $\{a, c\}$ could be added to G and embedded planarly such that it subdivides f. The resulting embedding would be NIC-planar, a contradiction to the maximality of G. Thus, $\{a, c\} \in E$, but it is not part of the boundary of f. By the same reasoning, if $\{b, d\} \notin E$ then it could be added. Hence, $\{a,c\},\{b,d\} \in E$. As neither of both edges subdivides f, $\{a,c\}$ and $\{b,d\}$ must cross each other in $\mathcal{E}(G)$ and thus, by Lemma 1, $\mathcal{E}(G)$ contains a corresponding kite of the K_4 induced by a,b,c,d. In particular, $\{a,d\} \in E$ (cf. Fig. 2a). Note that $\{a,d\}$ does not necessarily bound f. Nevertheless, this kite can be transformed into a tetrahedron by re-routing one of $\{a,c\}$ or $\{b,d\}$ such that it subdivides f.

W. l. o. g., consider the embedding $\mathcal{E}'(G)$ obtained from $\mathcal{E}(G)$ by re-routing $\{a,c\}$. Denote by f' and f'' the faces resulting from the division of f and let f' be the face bounded by $\{a,b\},\{b,c\},\{a,c\}$. In $\mathcal{E}'(G)$, the edges $\{a,c\},\{c,d\},\{a,d\}$ form a closed planar path p and thus split the set of faces into two partitions P' and P'', such that P' contains exactly the two triangles emerging from the conflation of the former kite faces as well as the face f' (cf. Fig. 2b). Note that only the vertices a,b,c,d are incident to faces in P'. If $n \geq 5$, there must be a fifth vertex, which is subsequently incident only to faces contained in P''. In particular, this vertex must be incident to at least two faces in P'' due to biconnectivity. This implies that P'' cannot consist only of face f'', and consequently, f'' cannot be bounded by the edge $\{a,d\}$. Recall that f'' results from the subdivision of face f in $\mathcal{E}(G)$ and that f's boundary consists only of planar edges. Hence, there is a vertex e incident to f'' such that f'' is bounded by $\{d,e\}$ and e is distinct from a,b,c,d. Observe that p contains c, but not e, and, by construction, f'' is the only face c is incident to in P''. As p is planar and the boundary of f'' contains $\{a,c\}$, $\{c,d\}$, and $\{d,e\}$, c and e cannot be adjacent in G. However, $\{c,e\}$ can be added to G and embedded planarly such that it subdivides f'', which yields a NIC-planar embedding, a contradiction to the maximality of G. □

Observe that the lemma neither holds for maximal 1-planar graphs nor for plane-maximal 1-planar graphs, as both admit hermits, i. e., vertices of degree two [16]. Moreover, the restriction to graphs with $n \geq 5$ is indispensable, as K_4 can be embedded as a kite, which has a quadrangular (outer) face. Forthcoming, we assume that all graphs have size $n \geq 5$.

Next, consider the planar versions of an embedding $\mathcal{E}(G)$ of a maximal NIC-planar graph G.

Corollary 1. *Let G be a maximal NIC-planar graph with $n \geq 5$.*

- *If $\widehat{G}_{\mathcal{E}} \subseteq G$ is a planar reduction of G with respect to any NIC-planar embedding $\mathcal{E}(G)$, then $\widehat{G}_{\mathcal{E}}$ is maximal planar.*
- *G is triconnected.*
- *The planar skeleton $\widetilde{G}_{\mathcal{E}} \subseteq G$ with respect to every NIC-planar embedding $\mathcal{E}(G)$ is triconnected.*

Proof. $\widehat{G}_{\mathcal{E}}$ is planar and triangulated by Lemma 2. Thus, it is triconnected and so is G as its supergraph. Then, also the planar skeleton $\widetilde{G}_{\mathcal{E}}$ is triconnected as shown by Alam et al. [2]. □

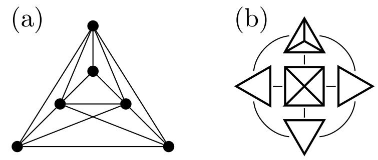

Figure 3: An embedding of a graph (a) and the corresponding generalized dual (b).

These results enable us to define the *generalized dual graph* of a maximal NIC-planar graph. As in the case of planar graphs, the dual is defined with respect to a specific embedding $\mathcal{E}(G)$. Figure 3 provides a small example.

Definition 1. *The* generalized dual graph $G^* = (V^*, E^*)$ *of a maximal* NIC*-planar graph* G *with respect to* NIC*-planar embedding* $\mathcal{E}(G)$ *contains three types of nodes: For every set of faces forming a kite in* $\mathcal{E}(G)$*,* V^* *contains a* $\boxtimes$*-node. For every set of faces forming a simple tetrahedron in* $\mathcal{E}(G)$*,* V^* *contains a* ◬*-node. All other faces of* $\mathcal{E}(G)$ *are represented by a* $\triangle$*-node each. Let* $q \in V^*$ *and denote by* $\mathcal{P}(q)$ *the set of faces of* $\mathcal{E}(G)$ *represented by* q*. As in conventional duals of planar graphs, there is an edge* $\{q, r\} \in E^*$ *for every pair of adjacent faces* $f \,|\, g$ *of* $\mathcal{E}(G)$ *such that* $f \in \mathcal{P}(q)$ *and* $g \in \mathcal{P}(r)$ *and* $q \neq r$*.*

In the following, we analyze the structure of G^*. For clarification, we call the elements of V vertices and the elements of V^* nodes. Note that $\boxtimes$-nodes have degree four and ◬- and $\triangle$-nodes have degree three. Furthermore, this definition in general allows for multi-edges, but not loops.

Lemma 3. *The generalized dual graph* G^* *of a maximal* NIC*-planar graph* G *with respect to a* NIC*-planar embedding* $\mathcal{E}(G)$ *is a simple* 3*-connected planar graph.*

Proof. Let $G' \subseteq G$ be the graph obtained from G by removing all vertices of degree three in G. Recall that every face in $\mathcal{E}(G)$ is a triangle by Lemma 2. Let $\mathcal{E}'(G')$ denote the NIC-planar embedding of G' inherited from $\mathcal{E}(G)$ and observe that $\mathcal{E}'(G')$ emerges from $\mathcal{E}(G)$ by removing the center vertex of each simple tetrahedron and thereby replacing every set of faces forming a simple tetrahedron embedding by a trivial triangle. Next, consider the planar skeleton $\widetilde{G'}_{\mathcal{E}'}$ of G' with respect to $\mathcal{E}'(G')$, which is obtained by removing all pairs of crossing edges in kites. Then the generalized dual graph G^* of G with respect to $\mathcal{E}(G)$ is the planar dual of $\widetilde{G'}_{\mathcal{E}'}$. As $\widetilde{G'}_{\mathcal{E}'}$ is simple and triconnected by Corollary 1, so is its dual graph. □

We say that a ◬- or $\triangle$-node $q \in V^*$ is *marked* if q is adjacent to a $\boxtimes$-node in G^*. Otherwise, q is *unmarked*. Two adjacent $\triangle$-nodes $q, r \in V^*$ are said to be *tetrahedral* if the union of their vertices induce K_4 in G. Observe that this induced K_4 is necessarily embedded as a non-simple tetrahedron. Let u, v, w and u, v, x denote the vertices incident to the faces represented by q and r, respectively. Then, $\{w, x\} \in E$ if and only if q, r are tetrahedral. In this case,

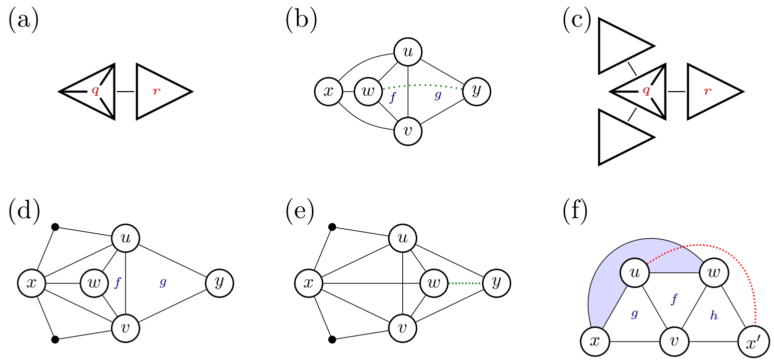

Figure 4: Proof of Lemma 4:
Case ii: A ◬-node q in G^* that is adjacent to an unmarked △-node r (a) and the corresponding situation in $\mathcal{E}(G)$, where $f \in \mathcal{P}(q)$ and $\mathcal{P}(r) = \{g\}$ (b).
Case (iii): A ◬-node q in G^* that is adjacent to a △-node r as well as two further △-nodes (c), the corresponding situation in $\mathcal{E}(G)$, where $f \in \mathcal{P}(q)$ and $\mathcal{P}(r) = \{g\}$ (d), and the reembedding, which then allows for the insertion of an additional edge (e).
Case (iv): The situation in $\mathcal{E}(G)$ in case of two adjacent, unmarked △-nodes q, r in G^* with $\mathcal{P}(q) = \{f\}$ and $\mathcal{P}(r) = \{g\}$ (f). The shaded region contains further vertices and edges of the graph.

we call $\{w, x\}$ the *tetrahedral edge*. In Fig. 4f, e. g., the △-nodes representing the faces f and g in G^* are tetrahedral and $\{w, x\}$ is the corresponding tetrahedral edge.

The definition of a NIC-planar embedding implies a number of restrictions on the adjacencies among nodes in G^*, which are subsumed in the following lemma.

Lemma 4. *Let G^* be the generalized dual of a maximal NIC-planar graph G where $n \geq 5$ with respect to a NIC-planar embedding $\mathcal{E}(G)$.*

(i) No two ⊠-nodes are adjacent.

(ii) A ◬-node is only adjacent to ⊠-nodes and marked △-nodes.

(iii) Every ◬-node is marked.

(iv) If two unmarked △-nodes are adjacent, then they are tetrahedral.

(v) If a △-node is adjacent to two △-nodes, then one of them is marked.

Proof. Consider a node q of G^* that is adjacent to another node r via an edge e. Let $\{u, v\}$ be the corresponding primal edge of e and denote by $f \in \mathcal{P}(q)$ and $g \in \mathcal{P}(r)$ the faces bounded by $\{u, v\}$. Note that $\{u, v\}$ is planar in $\mathcal{E}(G)$.

(i) Assume that q and r are ⊠-nodes. Then, u and v would be incident to two pairs of crossing edges, thus contradicting the NIC-planarity of $\mathcal{E}(G)$.

(ii) Next, assume that q is a ◬-node. Let $w, x \neq u, v$ denote the two other vertices of the K_4 represented by q, such that w is the vertex of degree three in the center of the tetrahedron.

Suppose that r is also a ⟁-node. As f and g are trivial triangles, u and v are not incident to a common pair of crossing edges by Lemma 1. Let y denote the third vertex incident to g, which is the center of the tetrahedron embedding represented by r. Then, w and y are not adjacent to each other, and no pair of vertices from u, v, w, y is incident to a common pair of crossing edges. In consequence, inserting an edge $\{w, y\}$ and embedding it such that it crosses $\{u, v\}$ does not violate NIC-planarity and thus contradicts the maximality of G. Subsequently, r cannot be a ⟁-node.

Suppose that r is a $\triangle$-node (cf. Fig. 4a and 4b). By Lemma 1, u and v cannot be adjacent to a common pair of crossing edges, as both f and g are trivial triangles. Let $y \neq u, v$ be the third vertex incident to g. If $x = y$, then G^* would consist of exactly one $\triangle$-node and one ⟁-node, i. e., G is K_4 and $\mathcal{E}(G)$ is its planar embedding. As $n \geq 5$, $x \neq y$. Being the tetrahedron's center vertex, w is adjacent only to u, v, and x and all edges incident to w are planar. Suppose that r is unmarked. Then, by Lemma 1, neither u, y nor v, y are adjacent to a common pair of crossing edges. Subsequently, the edge $\{w, y\}$ can be added to G and embedded such that it crosses $\{u, v\}$ without violating NIC-planarity, a contradiction to G being maximal. Hence, if a ⟁-node is adjacent to a $\triangle$-node, the latter must be marked.

(iii) Suppose that q is a ⟁-node as in (ii) and only adjacent to (marked) $\triangle$-nodes (cf. Fig. 4c and 4d). Then, in particular, r is a $\triangle$-node. Denote the third vertex incident to g by y. As $\{u, v\}$, $\{v, x\}$, and $\{u, x\}$ are planar and each of them bounds two trivial triangles, none of their end vertices can be incident to a common pair of crossing edges by Lemma 1. The same holds for u, w as well as v, w and x, w, since w is incident only planar edges. Subsequently, the embedding $\mathcal{E}'(G)$ obtained from $\mathcal{E}(G)$ by reembedding w such that $\{w, x\}$ crosses $\{u, v\}$ yields a NIC-planar embedding of G (cf. Fig. 4e). Furthermore, $\mathcal{E}'(G)$ contains a face that is bounded by four trivial edge segments $\{u, w\}$, $\{v, w\}$, $\{v, y\}$, and $\{u, y\}$. Thus, there also is a NIC-planar embedding for the graph obtained from G by adding an edge $\{w, y\}$, a contradiction to G being maximal. Note that as $n \geq 5$, $x \neq y$, hence, $\{w, y\}$ is not contained in G. In consequence, the ⟁-node q cannot be adjacent to $\triangle$-nodes only and thus must itself be marked.

(iv) Consider now the case that q is a $\triangle$-node and let w denote the third vertex incident to f. Furthermore, assume that r is another $\triangle$-node and both are unmarked (cf. Fig. 4f). Denote the third vertex incident to g by x. In consequence of Lemma 1, no pair of u, v, w is incident to a common pair of crossing edges, and likewise for u, v, x. Suppose that $\{w, x\} \notin E$. Then, by Lemma 1, w and x cannot be incident to a common pair of crossing edges. Hence, $\{w, x\}$ can be added to G and embedded such that it crosses $\{u, v\}$ without violating NIC-planarity, a contradiction to the maximality of G. Subsequently, $\{w, x\} \in E$. This in turn implies that $G[u, v, w, x]$ is K_4, hence, q and r are tetrahedral.

(v) Finally, consider the case that q and r are $\triangle$-nodes and that q is adjacent to a second $\triangle$-node s representing a face h, which is also depicted in Fig. 4f.

W. l. o. g., let $\{v, w\}$ be the edge bounding both f and h and denote by $x' \neq v, w$ the third vertex incident to h. Suppose that q, r, and s are unmarked. As we have argued in the proof of (iv), $\{w, x\} \in E$. For q and s, this analogously implies that $\{u, x'\} \in E$. As $\{w, x\}$, $\{x, v\}$, and $\{v, w\}$ form a closed path and both $\{w, x\}$ and $\{x, v\}$ are planar, $\{u, x'\}$ must cross $\{w, x\}$. By Lemma 1, $G[u, w, x, x']$ is K_4 and $\{u, w\}$ hence bounds a non-trivial triangle. Thus, q, r, and s are marked, a contradiction. Consequently, q, r, and s cannot be all unmarked. □

With respect to a fixed embedding, also the converse of Lemma 4 holds:

Lemma 5. *Let G^* be a triconnected planar graph with vertices of degree three that are labeled by △ and ◬ and of vertices of degree four that are labeled by ⊠ so that the requirements of Lemma 4 hold. Let G be a 1-planar graph whose generalized dual graph is G^*. Then, G is simple and triconnected and has a maximal NIC-planar embedding.*

Proof. A simple triconnected planar graph H has a simple triconnected dual H^* and H is isomorphic to the dual of H^*. Both graphs have a unique planar embedding.

Let G_1 be the planar dual of G^*. Then G_1 is simple and triconnected and has a unique planar embedding. First, add a pair of crossing edges in each quadrilateral face of G_1. This is the expansion of each ⊠-node. It preserves simplicity since otherwise G_1 were not triconnected. The so obtained graph G_2 is 1-planar. Next, expand each ◬-node by inserting a center vertex in the respective triangle and call the resulting graph G_3. This preserves simplicity and triconnectivity. The embedding $\mathcal{E}(G_3)$ is inherited from the embedding of G_1 and is triangulated and 1-planar. It is also NIC-planar, since two kites are not adjacent by requirement (i) of Lemma 4. It remains to show that $\mathcal{E}(G_3)$ is maximal NIC-planar. Towards a contradiction, suppose an edge $\{u, v\}$ could be added to $\mathcal{E}(G_3)$. Then u and v are in a face f of $\mathcal{E}(G_1)$ or in two adjacent faces f_1, f_2 with a common edge $\{a, b\}$ where the faces may be expanded by a center. Face f must be a quadrilateral, which, however, is expanded to a kite so that $\{u, v\}$ already exists in G_3. If u is the center of a ◬-node, then the given requirements from Lemma 4 exclude a new edge, since the other face is marked and the new edge would introduce two adjacent kites. If u and v are the vertices on opposite sides of two triangles with a common edge $\{a, b\}$, then $\{u, v\}$ is excluded by the requirements. G is isomorphic to G_3, since G^* is obtained from each as a generalized dual. □

Let $q \in V^*$ be a ⊠-node in G^* and $r, s \in V^*$ two △-nodes such that r, s are tetrahedral and the corresponding tetrahedral edge is one of the crossing edges of the kite represented by q. A *kite flip* between q, r, and s is a reembedding of the tetrahedral edge such that it crosses the edge on the common boundary of the faces represented by r and s. In the generalized dual $G^{*'}$ with respect to this new embedding $\mathcal{E}'(G)$, q is hence replaced by a pair of adjacent, tetrahedral △-nodes and r and s are replaced by a ⊠-node. More formally, if q is adjacent to

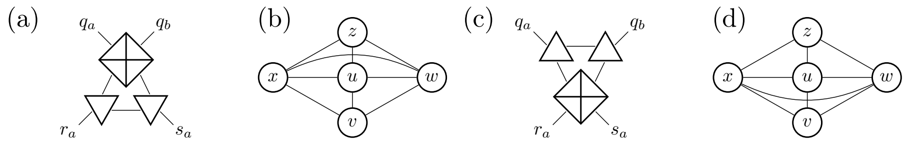

Figure 5: Example of a kite flip between a $\boxtimes$-node q and two $\triangle$-nodes r and s, where r and s are additionally adjacent to q. The adjacencies in the generalized dual graph are shown before (a) and after (c) the flip, the corresponding embeddings are depicted in (b) and (d).

nodes q_a, q_b, q_c, q_d, r is adjacent to s, r_a, and r_b, and s is additionally adjacent to s_a, and s_b, then $G^{*'}$ has the same set of vertices and edges as G^* except for q, r, and s and their incident edges. Instead, $G^{*'}$ contains two adjacent, tetrahedral $\triangle$-nodes q' and q'' and a $\boxtimes$-node t_{rs} such that, w. l. o. g., q' is adjacent to q_a and q_b, q'' is adjacent to q_c and q_d, and t_{rs} is adjacent to r_a, r_b, s_a, and s_b in $G^{*'}$. Fig. 5 provides an example, where the $\boxtimes$-node is even adjacent to the $\triangle$-nodes it is flipped with, which is, however, not a prerequisite.

In general, the embedding resulting from a kite flip in a (maximal) NIC-planar embedding is not necessarily also maximal or even NIC-planar.

Lemma 6. *Let G^* be the generalized dual of a maximal NIC-planar graph $G = (V, E)$ where $n \geq 5$ with respect to a NIC-planar embedding $\mathcal{E}(G)$. For every pair of adjacent $\triangle$-nodes in G^* that are either unmarked or adjacent to a single, common $\boxtimes$-node there is a maximal NIC-planar embedding $\mathcal{E}'(G)$ of G such that in the corresponding generalized dual, these two $\triangle$-nodes are kite flipped with a $\boxtimes$-node.*

Proof. Consider a pair of $\triangle$-nodes q and r in G^* that are adjacent to each other via an edge e. Let $\{u, v\}$ be the corresponding primal edge of e and denote by f and g the trivial triangles they represent, i. e., $\mathcal{P}(q) = \{f\}$ and $\mathcal{P}(r) = \{g\}$. Note that f and g are both bounded by $\{u, v\}$ and let w and x, $w \neq x$, denote the third vertex incident to f and g, respectively.

Assume first that q and r are unmarked, as obtained, e. g., in case of an embedding as in Fig. 6a. By Lemma 4, q and r are tetrahedral. Hence, $\{w, x\} \in E$ and is the corresponding tetrahedral edge. Suppose that w and x are not incident to a common pair of crossing edges. By Lemma 1 and Lemma 2, $\{w, x\}$ thus bounds two trivial triangles. Then, however, reembedding $\{w, x\}$ such that it crosses $\{u, v\}$ yields a NIC-planar embedding with a face whose boundary consists of four trivial edge segments, a contradiction to either Lemma 2 or the maximality of G. Hence, w and x must be incident to a common pair of crossing edges, i. e., G contains a K_4 induced by w, x and two further vertices y, z. Let $\mathcal{E}'(G)$ be the NIC-planar embedding obtained from $\mathcal{E}(G)$ by reembedding $\{w, x\}$ such that it crosses $\{u, v\}$ (Fig. 6b). As G is maximal, so must be $\mathcal{E}'(G)$. The reembeddability of $\{w, x\}$ together with Lemma 1 implies that $\{w, x\}$ is crossed in both $\mathcal{E}(G)$ and $\mathcal{E}'(G)$. Consequently, the generalized dual $G^{*'}$ with respect to $\mathcal{E}'(G)$ is obtained from G^* by replacing q and r with a $\boxtimes$-node and by replacing the $\boxtimes$-node representing the kite $G[\{w, x, y, z\}]$ with two $\triangle$-nodes, i. e., $G^{*'}$ is

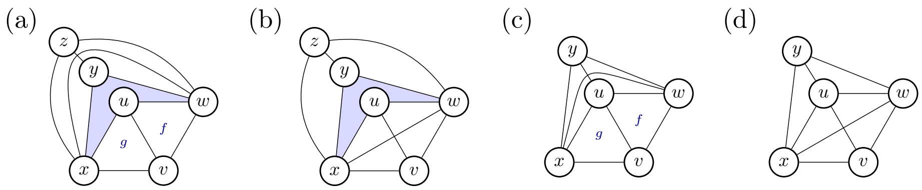

Figure 6: Proof of Lemma 6: If the △-nodes representing f and g are unmarked (a), the embedding obtained by a kite flip (b) is NIC-planar. The shaded region necessarily contains further vertices and edges. Likewise, if the △-nodes representing f and g are adjacent to a single, common ⊠-node (c), the embedding obtained by a kite flip (d) is NIC-planar.

obtained by a kite flip between this ⊠-node and q and r. Note that $y, z \neq u, v$, otherwise, q and r were adjacent to the ⊠-node representing the embedding of $G[\{w, x, y, z\}]$ and therefore marked.

Otherwise, assume that q and r are adjacent to a common ⊠-node s in G^*. Then, the faces represented by s are incident to both w and x as well as either u or v. W. l. o. g., assume the former and let y denote the fourth vertex of the K_4 represented by s, i. e., s represents the embedding of the K_4 $G[u, w, x, y]$. Figure 6c shows an embedding that corresponds to this situation in G^*. Subsequently, G contains the edge $\{w, x\}$, which implies that q and r are tetrahedral. As neither q nor r is adjacent to a further ⊠-node in G^* by the requirements of the lemma, $\{v, x\}$ and $\{v, w\}$ do not bound a non-trivial triangle face. Hence, neither u, v nor v, x nor v, w are adjacent to a common pair of crossing edges by Lemma 1. Furthermore, u, w and u, x are adjacent only to the pair of crossing edges $\{u, y\}$ and $\{w, x\}$. Let $\mathcal{E}'(G)$ be the NIC-planar embedding obtained from $\mathcal{E}(G)$ by reembedding $\{w, x\}$ such that it crosses $\{u, v\}$, i. e., by applying a kite flip to s, q and r (Fig. 6d). As $\{u, y\}$ and $\{w, x\}$ do not cross in $\mathcal{E}'(G)$, u, w and u, x are now again only adjacent to one common pair of crossing edges each, which is in both cases $\{u, v\}$ and $\{w, x\}$. Thus, $\mathcal{E}'(G)$ is a NIC-planar embedding of G, and, as G is maximal, so is $\mathcal{E}'(G)$. □

The requirements of Lemma 4 characterize a NIC-planar embedding $\mathcal{E}(G)$ of a maximal NIC-planar graph G. They guarantee that $\mathcal{E}(G)$ is maximal, but the graph G may still have another, non-maximal NIC-planar embedding, as Fig. 7 shows. Note that the second embedding emerges from a kite flip between the ⊠-node representing the kite embedding of the K_4 induced by a, b, and both red vertices and the two adjacent △-nodes representing the trivial triangle faces that e, one red vertex and either b or the second red vertex are incident to.

Conjecture 1. *If G is a graph with a maximal NIC-planar embedding $\mathcal{E}(G)$ that complies with Lemma 4 and every sequence of kite flips of a pair of adjacent △-nodes that are either unmarked or adjacent to a single, common ⊠-node yields in turn a maximal NIC-planar embedding, then G is maximal NIC-planar.*

Even though the generalized dual has a unique planar embedding, the previous results already demonstrated that this does not equally apply to maximal

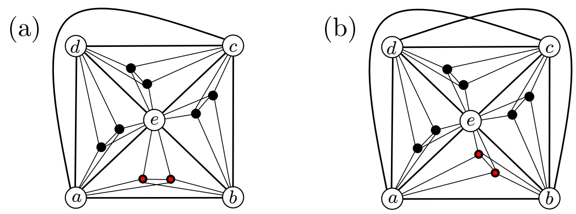

Figure 7: The embedding in (a) is maximal NIC-planar, however, re-embedding the K_5 subgraph with the vertices a, b, e admits the addition of the edge $\{b, d\}$ in the outer face, which yields the maximal NIC-planar graph in (b).

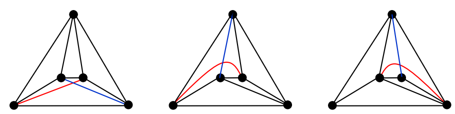

Figure 8: Three embeddings of K_5 with a fixed outer face. Each kite includes the edge between the inner vertices and one of the outer edges.

NIC-planar graphs in general. To obtain a generalized dual, an embedding is needed, which in particular also determines the pairs of crossing edges. In fact, even a maximal NIC-planar graph may admit many embeddings. To begin with, consider the complete graph K_5. It has one 1-planar embedding up to isomorphism [31] and admits three 1-planar and even IC-planar embeddings if the outer face is fixed, see Fig. 8, which each use one outer edge in a kite. Next, we want to study common 1-planar embeddings of two K_5 graphs. In general, two subgraphs H and H' are said to be *k-sharing* if they have at least k common vertices. They *share a crossing* in an embedding $\mathcal{E}(G)$ of their common supergraph G if there are edges e of H and e' of H' that cross in $\mathcal{E}(G)$.

Lemma 7. *If two K_5 subgraphs π and π' of G share a crossing in a 1-planar embedding $\mathcal{E}(G)$, then they are 3-sharing and this bound is tight.*

Proof. Let $\pi = G[v_1, v_2, v_3, v_4, v_5]$ and $\pi' = G[u_1, u_2, u_3, u_4, u_5]$ and consider the embeddings $\mathcal{E}(\pi)$ and $\mathcal{E}(\pi')$ inherited from $\mathcal{E}(G)$. Both $\mathcal{E}(\pi)$ and $\mathcal{E}(\pi')$ are

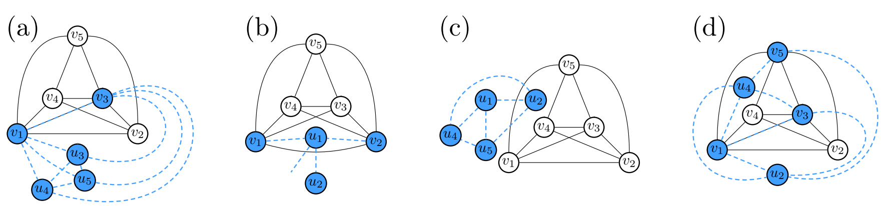

Figure 9: Two K_5s π and π' with a common edge that is crossed by another edge of π (a), an edge of π' crosses a planar kite edge of π (b), an edge of π' crosses a planar non-kite edge of π (c), and two 3-sharing K_5 subgraphs with a common 1-planar embedding (d). Black vertices and solid edges are those of π, colored vertices and dashed edges (also) belong to π'.

unique up to isomorphism, as depicted in Fig. 8. W. l. o. g., assume that $\{v_1, v_3\}$ crosses $\{v_2, v_4\}$ in $\mathcal{E}(\pi)$, i. e., $\kappa = G[v_1, v_2, v_3, v_4]$ is embedded as a kite.

Suppose that one of $\{v_1, v_3\}$ and $\{v_2, v_4\}$ is also an edge of π' (see Fig. 9a). This in particular includes the cases that the shared crossing involves one of $\{v_1, v_3\}$ or $\{v_2, v_4\}$ and that the edge of π' that crosses an edge of π is part of π. W. l. o. g., assume that $u_1 = v_1$ and $u_2 = v_3$. If π and π' are at most 2-sharing, there must be three further vertices u_3, u_4, u_5 of π' that are not part of π. Note that u_3, u_4, u_5 form a K_3 τ. As neither of $\{v_1, v_3\}$ and $\{v_2, v_4\}$ may be crossed a second time and $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_1\}$ may be crossed at most once, none of τ's vertices can be incident to a face of κ. Hence, they must be incident to one or more faces that are also incident to v_5, i. e., one of the trivial triangle faces of $\mathcal{E}(\pi)$. Due to 1-planarity and v_5 having vertex degree four, τ's vertices must all reside in the same face. Then, however, either one of τ's is crossed twice by edges incident to v_5 or vice versa, a contradiction to the 1-planarity of $\mathcal{E}(G)$.

Suppose that a planar edge of $\mathcal{E}(\kappa)$, w. l. o. g., $\{v_1, v_2\}$, is crossed by an edge $\{u_1, u_2\}$ of π', where, again w. l. o. g., u_1 is the vertex in one of the non-trivial triangle faces of κ (see Fig. 9b). Then, besides u_2, u_1 can be adjacent to at most v_1 and v_2, but not to any fifth vertex of π', irrespective of whether this vertex is also in π or not.

Finally, suppose that one of the edges incident to v_5, w. l. o. g., $\{v_1, v_5\}$ is crossed by an edge $\{u_1, u_2\}$ of π'. This situation is depicted in Fig. 9c. Note that the case where $\{u_1, u_2\}$ is also an edge of π has already been considered above. Hence, we can assume w. l. o. g., that u_1 is not a vertex of π. If π and π' are at most 2-sharing, there must be again at least two further vertices u_4 and u_5 that are adjacent to each other as well as to u_1 and u_2. In particular, every triple of u_1, u_2, u_4, and u_5 forms a K_3. By the above argument, if the vertices of such a K_3 are incident to a trivial triangle face of $\mathcal{E}(\pi)$, they must all reside in the same face. As $\{u_1, u_2\}$ crosses $\{v_1, v_5\}$, this does not apply to the two K_3s containing both u_1 and u_2, a contradiction. Note that our argumentation does not exclude the possibility that $u_2 = v_4$ or that v_1 and/or v_5 are also vertices of π'. However, if π' contains v_4, v_1, and v_5, then π and π' are 3-sharing.

Fig. 9d shows that if π and π' are 3-sharing, they may share a crossing in $\mathcal{E}(G)$, which testifies that this bound is tight. □

Note that the converse of Lemma 7 is not true. Any trivial triangle of an embedded K_5 can contain two additional vertices such that these plus the three triangle vertices form a second K_5. We will now use this result to prove an exponential number of embeddings.

Lemma 8. *There are maximal NIC-planar graphs with an exponential number of NIC-planar embeddings.*

Proof. We construct graphs G_k in two stages. First, consider the nested triangle graph T_k [22, 25] with vertices u_i, v_i, w_i at the i-th layer for $i = 1, \ldots, k$ in counter-clockwise order. There are *layer edges* $\{u_i, v_i\}, \{v_i, w_i\}, \{w_i, u_i\}$ for $i =$

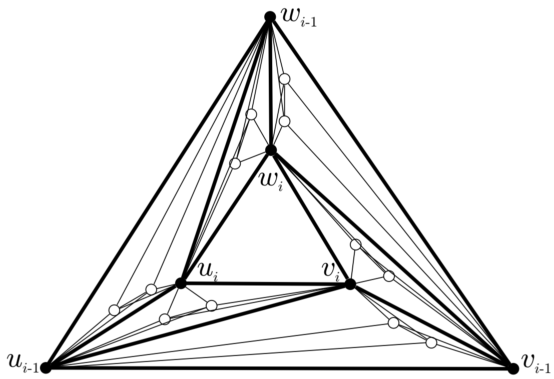

Figure 10: The layer i of G_k.

$1, \ldots, k$ and *intra-layer edges* $\{u_i, u_{i-1}\}, \{v_i, v_{i-1}\}, \{w_i, w_{i-1}\}$ and $\{u_i, w_{i-1}\}$, $\{v_i, u_{i-1}\}$, $\{w_i, v_{i-1}\}$ for $1 \leq i < k$, which triangulate T_k by planar edges.

For every triangle t with vertices u, v, w we insert two vertices a and b and add edges $\{u, a\}, \{u, b\}, \{v, a\}, \{v, b\}, \{w, a\}, \{w, b\}$ such that these five vertices together induce K_5. Fig. 10 shows one layer of G_k.

As every pair of K_5s in G_k is 2-sharing and every edge is part of at least one K_5, the edges shared by two K_5s are edges of T_k and embedded planar in $\mathcal{E}(G_k)$ by Lemma 7. Since $G_k[T_k]$ is triconnected it has a unique planar embedding. Every triangle t of $\mathcal{E}(T_k)$ includes two vertices such that there is a K_5. No further edge can be added without violating maximal NIC-planarity. Hence, G_k is maximal NIC-planar.

For each layer, there are at least two NIC-planar embeddings, one with the edges $\{u_i, u_{i-1}\}, \{v_i, v_{i-1}\}, \{w_i, w_{i-1}\}$ in kites and one with the edges $\{u_i, w_{i-1}\}$, $\{v_i, u_{i-1}\}, \{w_i, v_{i-1}\}$. In either case, the layer edges are not part of a kite so that the embeddings in the layers are independent. We thereby obtain at least 2^{k-1} NIC-planar embeddings. In order to distinguish identical embeddings up to a rotational symmetry, we fix the embedding of the K_4 in the first two layers so that at least 2^{k-3} different embeddings remain for graphs of size $15k-12$. $\square$

Planar embeddings have been generalized to maps [18] so that there is an adjacency between faces if their boundaries intersect. Also the intersection in a point suffices for an adjacency. There is a k-point p if k faces meet at p. A hole-free map $\mathcal{M}$ defines a hole-free graph G so that the faces of $\mathcal{M}$ correspond to the vertices of G and there is an edge if and only if the respective regions are adjacent. (There is a hole if a region is not associated with a vertex). Obviously, a k-point induces K_k as a subgraph of G. If no more than k regions meet at a point, then $\mathcal{M}$ is a k-map and G is a hole-free *k-map graph.* Chen et al. [18, 19] observed that a graph is triangulated 1-planar if and only if it is a 3-connected hole-free 4-map graph. For a triangulation it suffices that at least one embedding is triangulated. For NIC-planar graphs, this implies a characterization with an independent set.

Corollary 2. *A graph G is triangulated NIC-planar if and only if G is the graph of a hole-free 4-map $\mathcal{M}$ such that the 4-points of $\mathcal{M}$ are an independent set.*

Proof. Consider an embedding $\mathcal{E}(G)$ of a triangulated NIC-planar graph G and remove all pairs of crossing edges. Then, the planar dual of the obtained graph is a map $\mathcal{M}$ such that every 4-point corresponds one-to-one to a kite in $\mathcal{E}(G)$. The statement follows from the observation that two 4-points are adjacent in $\mathcal{M}$ if and only if two kites in $\mathcal{E}(G)$ share an edge. □

4. Density

For the analysis of densest and sparsest maximal NIC-planar graphs, we use the *discharging method* that was successfully applied for the proof of the 4-color theorem [4] and improving upper bounds on the density of quasi-planar graphs [1]. In this technique one assigns "charges" to the vertices and faces of a planar graph and computes the total charge after assigning all charges and after a redistribution (discharging phase).

We assign charges as follows: Consider the $\boxtimes$-nodes and their neighborhood in G^*. Let $\mathcal{E}(G)$ be an arbitrary NIC-planar embedding of a maximal NIC-planar graph G. Define the *level* $\mathcal{L}(q)$ of a node q of G^* as its minimum distance to a $\boxtimes$-node. Thus, every $\boxtimes$-node has level 0, and every $\triangle$- or ◬-node that is adjacent to a $\boxtimes$-node has level 1. A node with level x is also called an $\mathcal{L}_x$-node. Lemma 4 immediately implies:

Corollary 3. *Let G^* be the generalized dual with respect to a NIC-planar embedding $\mathcal{E}(G)$ of a maximal NIC-planar graph G with $n \geq 5$. Then, the level of a node is at most 2, every ◬-node has level 1 and can be adjacent only to $\mathcal{L}_0$- and $\mathcal{L}_1$-nodes, and every $\mathcal{L}_2$-$\triangle$-node is adjacent to at most one other $\mathcal{L}_2$-$\triangle$-node.*

Consider a $\boxtimes$-node q. Call the set of all nodes whose level equals their distance to q the *sphere* $\mathcal{S}(q)$ of q. Note that the spheres of two $\boxtimes$-nodes need not be disjoint (see, e. g., node t_1 in Fig. 11) and recall that a $\boxtimes$-node has four adjacencies. If we only consider a single adjacency and divide the $\triangle$- and ◬-nodes of a sphere equally among all spheres they belong to, we obtain pairwise disjoint *quarter spheres*. These quarter spheres are the elements of discharging. In consequence, we obtain fractions of nodes, e. g., $\frac{\triangle}{2}$- or $\frac{◬}{3}$-nodes. Fig. 11 depicts a clipping of the generalized dual of a graph. It shows three $\boxtimes$-nodes, whose spheres are indicated by the respective shaded regions. For the left, center, and right sphere, the patterns highlight one, three, and two quarter spheres, respectively. As no two $\boxtimes$-nodes can be adjacent, a quarter sphere can never be empty, but contains at least a fraction of a $\triangle$- or ◬-node. We denote the quarter sphere of q with respect to its neighbor r by $\mathcal{S}_Q(q, r)$.

Lemma 9. *Every quarter sphere of a $\boxtimes$-node q in the generalized dual of a maximal NIC-planar graph with respect to a NIC-planar embedding consists of at least either a $\frac{\triangle}{3}$- or a $\frac{◬}{3}$-node. It can at most be attributed either two $\triangle$-nodes, or a $\triangle$-, a $\frac{◬}{4}$-, and a $\frac{\triangle}{4}$-node, or a $\frac{◬}{2}$- and a $\frac{\triangle}{2}$-node.*

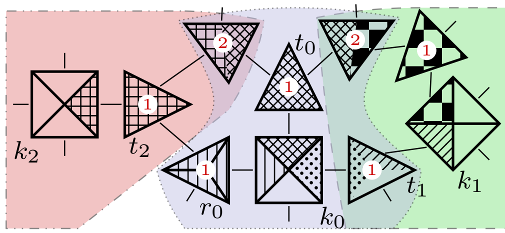

Figure 11: Clipping of a generalized dual graph, showing parts of the spheres (shaded) and quarter spheres (fill patterns) of three ⊠-nodes. The numbers correspond to the nodes' levels.

Proof. Observe that by definition, $\mathcal{S}(q)$ cannot contain another ⊠-node besides q and thus, neither can its quarter spheres. Consider a quarter sphere $\mathcal{S}_Q(q,r)$ of a ⊠-node q that is adjacent to a node r. Then, r is a ◬-node or a △-node.

We assume first that r is a ◬-node. Let $s \neq q$ and $t \neq q$ denote the two other neighbors of r besides q. By Corollary 3, $\mathcal{L}(s) \leq 1$ and $\mathcal{L}(t) \leq 1$. Thus, no part of them can be contained in $\mathcal{S}_Q(q,r)$, as their distance via r to a ⊠-node would be 2. In consequence of Lemma 4, neither s nor t is a ◬-node. If both are ⊠-nodes, r is shared among three spheres, if only one of them is a ⊠-node, it is shared among two spheres, and if both are △-nodes, it belongs entirely to $\mathcal{S}_Q(q,r)$. Hence, $\mathcal{S}_r(q)$ contains at least a $\frac{\text{◬}}{3}$-node.

For the upper bound, we assume that r is contained entirely in $\mathcal{S}_Q(q,r)$. Then, s and t must be △-nodes, which are in turn adjacent to at least one ⊠-node each. We split $\mathcal{S}_Q(q,r)$ once more into two *semi-quarter spheres* that each contain one half of r and, depending on their belonging to $\mathcal{S}_Q(q,r)$, the (fractions of) s and t, respectively. As G^* is simple and a node's maximum level is 2, the semi-quarter spheres are disjoint. W. l. o. g., consider s and let s' be a ⊠-node adjacent to s. Then, one of the semi-quarter spheres of $\mathcal{S}_Q(s',s)$ consists of exactly one $\frac{\triangle}{2}$-node. Hence, for every semi-quarter sphere containing a $\frac{\text{◬}}{2}$-node, there is at most one other semi-quarter sphere containing a $\frac{\triangle}{2}$-node. We can therefore attribute to each of both semi-quarter spheres a $\frac{\text{◬}}{4}$-node and a $\frac{\triangle}{4}$-node.

Assume now that r is a △-node. Then, due to Corollary 3, $\mathcal{S}_Q(q,r)$ cannot contain a ◬-node at all. As above, $\mathcal{S}_Q(q,r)$ must contain r with a proportion of at least $\frac{1}{3}$. This lower bound is reached if r is adjacent to three ⊠-nodes and hence shared among three quarter spheres.

We obtain a maximal quarter sphere if r is entirely part of $\mathcal{S}_Q(q,r)$. The case that r is adjacent to a ◬-node is already covered in the above argument from the viewpoint of the ◬-node. Thus, we only address the case where a △-node is adjacent to r. Consider again the semi-quarter spheres. By Corollary 3, r may be adjacent to a $\mathcal{L}_2$-△-node s. As a $\mathcal{L}_2$-△-node can be adjacent to at most one other $\mathcal{L}_2$-△-node, s must be adjacent to another $\mathcal{L}_1$-node besides r, i. e., s must be shared between at least two quarter spheres. Subsequently, the semi-quarter sphere consists of at most two $\frac{\triangle}{2}$-nodes.

As indicated above and depicted in Fig. 11, it is possible for a $\mathcal{L}_1$-△-node to be adjacent to both a $\mathcal{L}_2$-$\triangle$-node and a $\triangle\!\!\!\!\triangle$-node. Hence, we obtain the three stated combinations. □

The generalized dual graph introduced in the previous section enables us to estimate the density of maximal NIC-planar graphs.

Lemma 10. *Let G be a maximal NIC-planar graph. If G is optimal, then every quarter sphere with respect to any NIC-planar embedding of G consists of a $\frac{\triangle}{3}$-node and G has four planar edges for every pair of crossing edges. If G is sparsest maximal, then every quarter sphere can at most be attributed either two $\triangle$-nodes, or a $\triangle$-, a $\frac{\triangle\!\!\!\!\triangle}{4}$-, and a $\frac{\triangle}{4}$-node, or a $\frac{\triangle\!\!\!\!\triangle}{2}$- and a $\frac{\triangle}{2}$-node. For every pair of crossing edges, G has 14 planar edges.*

Proof. Let $\mathcal{E}(G)$ be a NIC-planar embedding of a maximal NIC-planar graph G and denote by G^* the corresponding generalized dual graph. Consider the quarter spheres of the $\boxtimes$-nodes in G^*. As every sphere contains exactly one $\boxtimes$-node and otherwise only nodes representing planar edges, we obtain a lower bound on the density of G if the quarter spheres' sizes are maximized and an upper bound if they are minimized. Recall that every edge segment separates two faces. Thus, we assign to every face only one half of the edges on its boundary. A $\triangle$-node represents one face which is a trivial triangle. Thus, every $\triangle$-node corresponds to $\frac{3}{2}$ planar edges. A $\triangle\!\!\!\!\triangle$-node represents three faces each of which is a trivial triangle and therefore corresponds to $\frac{9}{2}$ planar edges. The set of faces represented by a $\boxtimes$-node consists of four non-trivial triangles, which yields a pair of crossing edges and $\frac{4}{2}$ planar edges.

Due to Lemma 9, a quarter sphere contains at least $\frac{1}{3}$ of a $\triangle\!\!\!\!\triangle$-node or $\frac{1}{3}$ of a $\triangle$-node, which corresponds to $\frac{3}{2}$ and $\frac{1}{2}$ planar edges, respectively. The smaller the ratio of planar edges per crossing edge the larger the density of G. Thus, we obtain a minimum of $4 \cdot \frac{1}{2} + 2 = 4$ planar edges and one pair of crossing edges for an entire sphere, including the edges represented by the $\boxtimes$-node. On the other hand, a quarter sphere can be attributed at most either two $\triangle$-nodes or a $\triangle$-node, a $\frac{\triangle\!\!\!\!\triangle}{4}$-node, and a $\frac{\triangle}{4}$-node, or a $\frac{\triangle\!\!\!\!\triangle}{2}$-node and a $\frac{\triangle}{2}$-node—which corresponds to 3 planar edges in all three cases. For the entire sphere, this yields a total of $4 \cdot 3 + 2 = 14$ planar edges and one pair of crossing edges. □

In case of optimal NIC-planar graphs, Lemma 10 implies:

Corollary 4. *The generalized dual graph of every NIC-planar embedding of an optimal NIC-planar graph is bipartite with vertex set $V^*_{\boxtimes} \cup V^*_{\triangle}$ such that $V^*_{\boxtimes}$ contains only $\boxtimes$-nodes and $V^*_{\triangle}$ contains only $\triangle$-nodes.*

With respect to NIC-planar embeddings of optimal NIC-planar graphs, this unambiguity immediately yields:

Corollary 5. *The NIC-planar embedding of an optimal NIC-planar graph is unique up to isomorphism.*

Note that every optimal 1-planar graph has one, two, or eight 1-planar embeddings [36], which are again unique up to isomorphism.

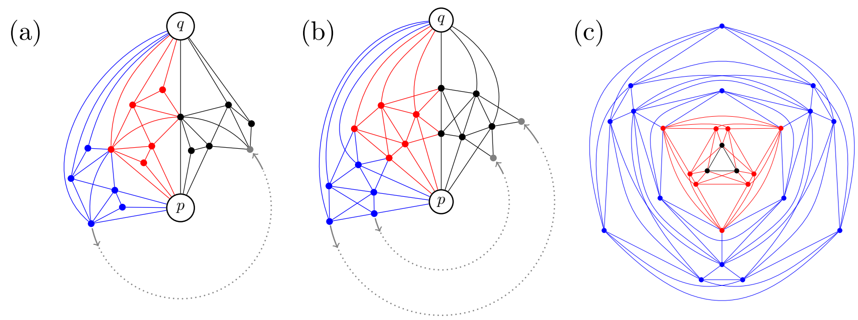

Figure 12: Construction of sparsest maximal (a) and optimal (b,c) NIC-planar graphs.

Lemma 10 suffices to prove upper and lower bounds on the density of maximal NIC-planar graphs. Whereas the upper bound of $18/5(n-2)$ edges was already proven by Zhang [38] and shown to be tight by Czap and Šugerek [21], the lower bound has never been assessed before.

Theorem 1. *Every maximal NIC-planar graph on n vertices with $n \geq 5$ has at least $\frac{16}{5}(n-2)$ and at most $\frac{18}{5}(n-2)$ edges. Both bounds are tight for infinitely many values of n.*

Proof. Let G be a maximal NIC-planar graph with $n \geq 5$ vertices and m edges. Let $\mathcal{E}(G)$ be a NIC-planar embedding of G and G^* be the corresponding generalized dual graph. Consider the planar subgraph $\widehat{G} \subseteq G$ that is obtained by removing exactly one of each pair of crossing edges. Then, $\widehat{G}$ has n vertices and $\widehat{m} \leq m$ edges. As $\widehat{m}$ is a triangulated planar graph, $\widehat{m} = 3n-6$.

By Lemma 10, there are at least 4 and at most 14 planar edges per pair of crossing edges. Furthermore, $\widehat{G}$ contains all planar edges as well as one edge from each pair of crossing edges. Hence, m and $\widehat{m}$ differ by at most $\frac{\widehat{m}}{5}$ and at least $\frac{\widehat{m}}{15}$. Thus, $\widehat{m} + \frac{\widehat{m}}{15} \leq m \leq \widehat{m} + \frac{\widehat{m}}{5}$, which yields with $\widehat{m} = 3n-6$ that $\frac{16}{5}n - \frac{32}{5} \leq m \leq \frac{18}{5}n - \frac{36}{5}$.

Fig. 12 shows how to construct a family of maximal NIC-planar graphs that meet the lower (Fig. 12a) and upper (Fig. 12b,c) bound exactly. In both cases, the blue subgraph can be copied arbitrarily often and attached either circularly (Fig. 12a,b) or to the outside (Fig. 12c). To obtain a sparsest maximal graph with $n = 7$, we take the red subgraph plus the leftmost black vertex in Fig. 12a and identify p and q. □

Corollary 6. *For every $k \geq 1$ there is a sparsest maximal NIC-planar graph with $n = 5k+2$ and $m = 16k$. There is an optimal NIC-planar graph with $n = 5k+2$ and $m = 18k$ if and only if $k \geq 2$.*

Proof. The constructions given in Fig. 12 show how to obtain sparsest and densest graphs with $n = 5k+2$. However, there is no optimal NIC-planar graph with 7 vertices and 18 edges: Such a graph must have $18 - (3 \cdot 7 - 6) = 3$ pairs of crossing edges. Consequently, any NIC-planar embedding must have three kites,

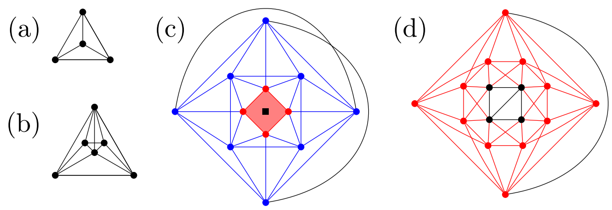

Figure 13: Construction of densest graphs with n vertices such that $n = 5k + 2 + 1$ (a), $n = 5k + 2 + 3$ (b), $n = 5k + 2 + 2$ and $n = 5k + 2 + 4$ (c,d).

which is impossible, as two kites in a NIC-planar embedding can share at most one vertex and need seven vertices. □

Theorem 2. *For every $i \in \{1, \dots, 4\}$ and infinitely many $k \geq 2$ there is a densest NIC-planar graph with $n = 5k + 2 + i$ and $m = \lfloor \frac{18}{5}(n-2) \rfloor$.*

Proof. For $i = 1$, a densest NIC-planar graph G_1 with $m = \lfloor \frac{18}{5}(n-2) \rfloor = \lfloor \frac{18}{5}(5k+1) \rfloor = 18k + 3$ edges can be obtained from any optimal NIC-planar graph G_0 by selecting any trivial triangle in any NIC-planar embedding of G_0 and inserting a vertex v along with three edges, each of which connects v to one of the three vertices of the triangle as depicted in Fig. 13a. In terms of the generalized dual, this corresponds to replacing one △-node by a ◬-node.

For $i = 2$, a densest NIC-planar graph G_2 with $m = \lfloor \frac{18}{5}(5k+2) \rfloor = 18k + 7$ edges can be constructed as depicted in Fig. 13c and Fig. 13d: First, we start with the graph shown in Fig. 13d, which consists of four black and twelve red vertices and five black and 44 red edges. We, however, do not add the black edge connecting two red vertices. Instead, we attach eight blue vertices and 28 blue edges as shown in Fig. 13c. This yields a graph with $24 = 5 \cdot 4 + 2 + 2$ vertices (i. e., $k = 4$) and 77 edges. The construction can be finished by adding the two black crossing edges that connect blue vertices as shown again in Fig. 13c. G_2 thus has 24 vertices and $79 = 18 \cdot 4 + 7$ edges. To obtain larger graphs, attach the twelve red vertices and 44 red edges shown in Fig. 13d and once more the blue subgraph of Fig. 13c. Repeat these two steps arbitrarily often before finally adding the two black crossing edges. For each repetition r, this yields another $12+8 = 20$ vertices and $44+28 = 72$ edges. Hence, each such graph has $24+20r$ vertices (i. e., $k = 4 + 4r$) and $79 + 72r = 18 \cdot (4 + 4r) + 7$ edges.

For $i = 3$, a densest NIC-planar graph G_3 with $m = \lfloor \frac{18}{5}(5k+3) \rfloor = 18k+10$ edges can again be obtained from any optimal NIC-planar graph G_0 by selecting any trivial triangle in any NIC-planar embedding of G_0 and inserting the subgraph depicted in Fig. 13b. More precisely, we add three vertices and ten edges such that the △-node in the corresponding generalized dual is replaced by a ⊠-node and four △-nodes.

Finally, for $i = 4$, a densest NIC-planar graph G_4 with $m = \lfloor \frac{18}{5}(5k+4) \rfloor = 18k + 14$ edges can be constructed similarly to the case for $i = 2$. We start again with the graph depicted in Fig. 13d, which has four black and twelve red vertices as well as five black and 44 red edges. Here, the construction can already be finished by adding the black edge connecting two red vertices, which yields a graph with $4 + 12 = 16 = 5 \cdot 2 + 2 + 4$ vertices (i. e., $k = 2$) and $5 + 44 + 1 = 50 = 18 \cdot 2 + 14$ edges. For a larger graph, we add again the blue subgraph with eight vertices and 28 edges as depicted in Fig. 13c as well as once more the red subgraph with twelve vertices and 44 edges arbitrarily often prior to connecting the two red vertices by the black edge. For each repetition r, this yields another $8 + 12 = 20$ vertices and $28 + 44 = 72$ edges. Hence, each such graph has $16 + 20r$ vertices (i. e., $k = 2 + 4r$) and $50 + 72r = 18 \cdot (2 + 4r) + 14$ edges.

Note that by Corollary 6, optimal graphs, which provide the basis for the construction in case of $i = 1$ and $i = 3$, only exist for $k \geq 2$ and that the graphs constructed for $i = 2$ and $i = 4$ have $k \geq 4$ and $k \geq 2$, respectively. □

From Corollary 6 and the proof of Theorem 2 we obtain:

Corollary 7. *There are densest NIC-planar graphs for all $n = 5k + 2 + i$ with $i \in \{0, 1, 3\}$ and all $k \geq 2$.*

Concerning IC-planar graphs, there are optimal ones with $\frac{13}{4}n - 6$ edges for all $n = 4k$ and $k \geq 2$ [39]. A densest IC-planar graph of size $n \geq 8$ is obtained from an optimal one of size $4k$ with $k = \lfloor \frac{n}{4} \rfloor$ by the replacement of i △-nodes by ◬-nodes, for $i = 1, 2, 3$. Each ◬-node adds one vertex and three edges. There are sparsest IC-planar graphs with $3n - 5$ edges for all $n \geq 5$, and this bound is tight [7].

5. Recognizing Optimal NIC-Planar Graphs in Linear Time

The study of maximal NIC-planar graphs in Sect. 3 also provides a key to a linear-time recognition algorithm for optimal NIC-planar graphs. Therefore, we establish a few more properties of optimal NIC-planar graphs and their embeddings $\mathcal{E}(G)$. An edge e is called k-fold K_4-*covered* in G if e is part of k K_4 subgraphs. We start with the following observation:

Lemma 11. *Let $\mathcal{E}(G)$ be a NIC-planar embedding of an optimal NIC-planar graph G. Every edge $e \in E$ is at least 1-fold K_4-covered and there is exactly one K_4 in G that contains e and is embedded as a kite in $\mathcal{E}(G)$.*

Proof. Corollary 4 implies that every edge $e \in E$ is at least once K_4-covered. Moreover, if e is contained in at least two different K_4 inducing subgraphs that are both embedded as a kite, then e's end vertices are incident to two common pairs of crossing edges, a contradiction to NIC-planarity. □

In Sect. 2 we already noted that in any NIC-planar embedding, every subgraph that induces K_4 must be embedded either as a kite or as a tetrahedron, which may in turn be simple or not. Recall that in case of a non-simple tetrahedron embedding, K_4's edges may not cross each other, but they may be crossed by edges that do not belong to this subgraph. The following lemma limits the possibilities of how the K_4 subgraphs and their embeddings can interact.

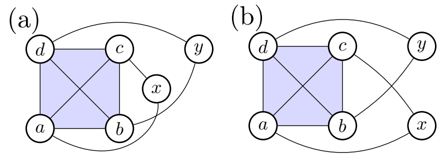

Figure 14: Proof of Lemma 12.

Lemma 12. *Let $\mathcal{E}(G)$ be a NIC-planar embedding of a maximal NIC-planar graph G and let $\{a, c\}$ and $\{b, d\}$ be two edges that cross each other in $\mathcal{E}(G)$. Then one of $\{a, c\}$ and $\{b, d\}$ is 1-fold K_4-covered.*

Proof. Let $\kappa = G[a, b, c, d]$. By Lemma 1, κ is K_4. Suppose for the sake of contradiction that there are subgraphs $\kappa' \neq \kappa$ and $\kappa'' \neq \kappa$ of G that both induce K_4 and such that κ' contains $\{a, c\}$ and κ'' contains $\{b, d\}$. Note that $\kappa' \neq \kappa''$, otherwise both had vertex set $\{a, b, c, d\}$ and hence, $\kappa = \kappa' = \kappa''$. Furthermore, both κ' and κ'' must be embedded as tetrahedrons, otherwise, a, c and b, d were incident to two pairs of crossing edges. Denote by $x \neq y \in V \setminus \{a, b, c, d\}$ two further vertices of G such that κ' contains x and κ'' contains y.

Consider the closed path (a, b, c, d) of planar edges, which partitions the set of faces of $\mathcal{E}(G)$ into P' and P''. Due to Lemma 2, one of these partitions, w. l. o. g. P', contains only the non-trivial triangles that form the kite embedding of κ. Hence, all faces incident to x and y must reside within P'' and the edges or edge segments of $\{a, x\}$, $\{c, x\}$, $\{b, y\}$, and $\{d, y\}$ only bound faces contained in P''. Consequently, the paths (a, x, c) and (b, y, d) must cross each other in P''.

Recall that a, b, c, and d are already pairwisely incident to a pair of crossing edges, namely $\{a, c\}$ and $\{b, d\}$. Fig. 14 shows two of the four possible pairs of additional crossing edges. Suppose that $\{a, x\}$ crosses $\{b, y\}$. Then, a and c are incident to another pair of crossing edges, a contradiction to the NIC-planarity of $\mathcal{E}(G)$. Likewise, if $\{c, x\}$ crosses $\{b, y\}$, or $\{a, x\}$ crosses $\{d, y\}$, or $\{c, x\}$ crosses $\{d, y\}$, then b and c, or a and d, or c and d, respectively, are incident to two common pairs of crossing edges, thereby again contradicting the NIC-planarity of $\mathcal{E}(G)$.

Subsequently, κ' and κ'' cannot both exist. □

The combination of Lemma 11 and Lemma 12 yields a characterization of those K_4 inducing subgraphs that are embedded as kite:

Corollary 8. *Let κ be a subgraph inducing K_4 in an optimal NIC-planar graph G and let $\mathcal{E}(G)$ be a NIC-planar embedding of G. Then, κ is embedded as a kite in $\mathcal{E}(G)$ if and only if κ has a 1-fold K_4-covered edge.*

Proof. Let $\{a, b, c, d\}$ denote the vertex set of κ. By Lemma 11, every edge of κ is at least once covered by a K_4 which is embedded as a kite.

If κ is embedded as a kite, then one of its crossing edges is 1-fold K_4-covered by Lemma 12, and if κ is embedded as a tetrahedron, then each of its edges is at least 2-fold K_4-covered. □

Algorithm 1 A recognition algorithm for optimal NIC-planar graphs.

Input: graph $G = (V, E)$ with $n = |V| \geq 5$ and $m = |E|$
Output: NIC-planar embedding $\mathcal{E}(G)$ if G is optimal NIC-planar, else $\perp$

1: **procedure** TestOptimalNIC(G)
2: **if** $m \neq \frac{18}{5}(n-2)$ **then return** $\perp$ ▷ G is not optimal.
3: $\mathcal{K} \leftarrow$ set of K_4s in G or $\perp$ in case of timeout
4: **if** $\mathcal{K} = \perp$ **then return** $\perp$
5: $\mathcal{K}_\times \leftarrow \emptyset$
6: create an empty bucket $B[e] = \emptyset$ for each edge $e \in G$
7: **for all** $\kappa \in \mathcal{K}$ **do**
8: add κ to every bucket $B[e]$ for every edge e of κ
9: **for all** $e \in E$ **do**
10: **if** $B[e] = \{\kappa\}$ **then**
11: $\mathcal{K}_\times \leftarrow \mathcal{K}_\times \cup \{\kappa\}$ ▷ κ must be embedded as kite by Corollary 8.
12: **for all** $e \in E$ **do**
13: **if** $|B[e] \cap \mathcal{K}_\times| \neq 1$ **then return** $\perp$ ▷ Lemma 11 is violated.
14: $G' \leftarrow G$
15: **for all** $\kappa \in \mathcal{K}_\times$ **do**
16: remove all edges of κ in G'
17: add a dummy vertex z_κ along with edges to all vertices of κ in G'
18: **if** G' is not planar **then return** $\perp$
19: $\mathcal{E}(G') \leftarrow$ planar embedding of G'
20: $\mathcal{E}(G) \leftarrow$ NIC-planar embedding of G obtained from $\mathcal{E}(G')$
21: **return** $\mathcal{E}(G)$

Now we are ready to proof the main result of this section.

Theorem 3. *There is a linear-time algorithm that decides whether a graph is optimal NIC-planar and, if positive, returns a NIC-planar embedding.*

Proof. Consider the algorithm given in Algorithm 1, which takes a graph G as input and either returns a NIC-planar embedding $\mathcal{E}(G)$ if G is optimal NIC-planar and otherwise returns $\perp$.

Let $G = (V, E)$. First, if the number of edges m of G does not meet the upper bound of $\frac{18}{5}(n-2)$, G cannot be optimal NIC-planar. The algorithm therefore returns $\perp$ in line 2 if this check fails. For the remainder of the algorithm, we can assume that $m \in \mathcal{O}(n)$.

Next, we identify those K_4 inducing subgraphs of G that must be embedded as a kite. To this end, enumerate all subgraphs of G that induce K_4 and keep them as set $\mathcal{K}$. This can be accomplished in linear time by running the algorithm of Chiba and Nishizeki [20] for at most $256n$ essential steps. A step is essential if it marks a vertex or an edge. The inessential steps, like unmark and print, take linear time in the number of essential steps. If G is maximal NIC-planar, then G has arboricity four [33] and the algorithm completes the computation of the K_4 listing within $256n$ essential steps. Otherwise, if the algorithm by Chiba and Nishizeki exceeds the bound on the running time, G does not have arboricity four and is hence not NIC-planar. Then our algorithm returns $\perp$ in line 4. Chen et al.[19] have shown that triangulated 1-planar graphs of size n have at most $27n$ K_4 subgraphs. The subset $\mathcal{K}_\times$ of $\mathcal{K}$ that will later contain those with kite embeddings is set to $\emptyset$. Initialize an empty bucket $B[e]$ for every edge $e \in E$. We now employ a variant of bucket sort on $\mathcal{K}$ and place a copy of every element $\kappa \in \mathcal{K}$ in all six buckets that represent an edge of κ. As the size of $\mathcal{K}$ is linear in the size of G, this takes $\mathcal{O}(n)$ time.

Afterwards, we loop over the edges of G and apply Corollary 8: If an edge is contained in exactly one $\kappa \in \mathcal{K}$, then κ must be embedded as a kite and is therefore added to $\mathcal{K}_\times$ in line 11. This can be accomplished in time $\mathcal{O}(m) = \mathcal{O}(n)$. Having identified the K_4 inducing subgraphs that must be embedded with a crossing, we can check whether every edge is contained in exactly one kite as required by Lemma 11 again in $\mathcal{O}(n)$ time.

The last step in the algorithm consists in identifying the pairs of crossing edges and, if possible, obtaining a NIC-planar embedding of G. For this purpose, we construct a graph G' from G as follows: For every $\kappa \in \mathcal{K}_\times$ with vertex set $\{a, b, c, d\}$, remove all edges connecting a, b, c, and d. Then, add a new dummy vertex z_κ along with edges $\{a, z_\kappa\}$, $\{b, z_\kappa\}$, $\{c, z_\kappa\}$, $\{d, z_\kappa\}$. Thus, κ is replaced by a star with center z_κ. This construction requires again $\mathcal{O}(n)$ time.

Observe that G' is a subgraph of every planarization of G with respect to any NIC-planar embedding of G. Hence, if G' is not planar, then G cannot be optimal NIC-planar, so the algorithm returns $\perp$ in line 18. Otherwise, we obtain a planar embedding $\mathcal{E}(G')$ of G. This can be done in time $\mathcal{O}(n)$.

Next, construct an embedding $\mathcal{E}(G)$ of G from $\mathcal{E}(G')$ by replacing every dummy vertex z_κ and its incident edges by the edges of κ. The edges incident to z_κ are taken as non-trivial edge segments and the remaining edges are routed close to these edges. This corresponds to replacing the four trivial triangles incident to z_κ in $\mathcal{E}(G')$ one-to-one by four non-trivial triangles forming the kite embedding of κ and takes again $\mathcal{O}(n)$ time. $\mathcal{E}(G)$ now is a NIC-planar embedding of G such that exactly the elements of $\mathcal{K}_\times$ are embedded with a crossing. As the number of edges in G meets the upper bound of $3.6(n-2)$ exactly, G is optimal NIC-planar and the algorithm returns $\mathcal{E}(G)$ as a witness. The overall running time is in $\mathcal{O}(n)$. □

6. Drawing NIC-Planar Graphs

Every NIC-planar graph is a subgraph of a maximal NIC-planar graph and every NIC-planar embedding with $n \geq 5$ has a trivial triangle which we use as outer face. By Corollary 1 maximal NIC-planar graphs are triconnected. Hence, the linear-time algorithm of Alam et al. [2] for 1-planar graphs can be used.

Corollary 9. *Every NIC-planar graph has a NIC-planar straight-line drawing on an integer grid of $\mathcal{O}(n^2)$ size.*

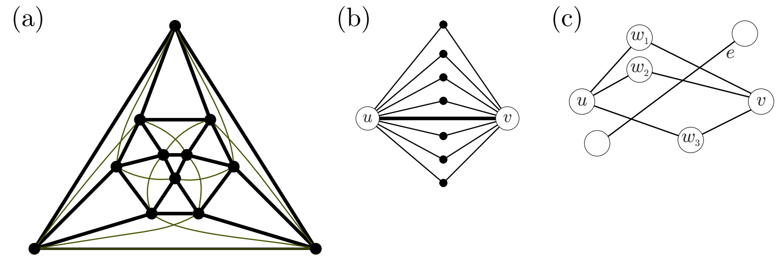

Figure 15: (a): The graph G. The supergraph G^+ extends G by seven 2-paths for each fat edge. G corresponds to the red and black part of Fig. 12c. (b): Seven 2-paths augment an edge to a fat edge. (c): A forced fan crossing.

A graph G has *geometric thickness* k if G if admits a straight-line drawing in the plane such that there is a k-coloring of the edges and edges with the same color do not cross [24].

Corollary 10. *Every NIC-planar graph has geometric thickness two.*

However, NIC-planar graphs do not necessarily admit straight-line drawings with right angle crossings. In consequence, the classes of NIC-planar graphs and RAC graphs are incomparable, since RAC graphs may be too dense [23].

Theorem 4. *There are infinitely many NIC-planar graphs that are not RAC graphs, and conversely.*

For the harder part, we construct a NIC-planar graph that is not RAC. Infinitely many graphs are obtained by multiple copies. Let $G^+ = (V^+, E^+)$ be obtained from graph $G = (V, E)$ in Fig. 15a by augmenting every planarly drawn edge between two vertices u and v with seven vertex-disjoint 2-paths, as shown in Fig. 15b. Every edge of G that is augmented is called a *fat edge*. If u and v are connected by a fat edge, then v is a *fat neighbor* of u. We show that graph G^+ does not admit a RAC drawing.

Observe that G consists of six K_4 and eight K_3 subgraphs such that a K_4 is attached to each side of a K_3. As shown in Fig. 15a, G is NIC-planar and likewise is G^+, since the 2-paths of each fat edge can be embedded planarly. We obtain the *induced* RAC drawing $\mathcal{D}(G)$ of G from a RAC drawing $\mathcal{D}(G^+)$ of G^+ by removing the 2-paths of each fat edge.

Graph G is 4-connected and G and G^+ remain biconnected if all pairs of crossing edges are removed and additionally either a single K_3 or a single K_4. After the removal, each vertex of G and G^+ still has two fat neighbors and the remaining graph is connected using only fat edges. The following properties are immediate (see Fig. 15c):

Lemma 13 ([23])**.** *A RAC drawing does not admit a* fan-crossing*, i. e., no edge may cross two edges with a common end vertex.*

Lemma 14. *If an edge is crossed by k 2-paths $p_i = (u, w_i, v)$ for $i = 1, \ldots, k$ connecting two vertices u and v in a RAC drawing, then $k \leq 2$.*

Lemma 15. *If there is a triangle τ in a RAC drawing and a fat edge $\{u, v\}$ so that u, v are not vertices of τ, then u and v are either both inside or both outside τ.*

Proof. Every vertex w_i of a 2-path (u, w_i, v) has degree 2 and thus cannot be a vertex of τ. If, w. l. o. g., u is inside and v is outside of τ, at least one edge of τ is crossed by at least three 2-paths, which contradicts Lemma 14. □

Lemma 16. *Let $\mathcal{D}(G^+)$ be a RAC drawing of G^+. Then every K_4 is drawn with a pair of crossing edges.*

Proof. Suppose that $G^+[\{u, v, w, x\}]$ is a K_4 which is not drawn with a pair of crossing edges. Then it is drawn as a tetrahedron [31]. W. l. o. g., let x be inside the triangle f_{uvw}. Then, f_{uvw} is partitioned into three triangles f_{uvx}, f_{uwx}, and f_{vwx}.

Vertex x has a fat neighbor $y \notin \{v, w, x\}$. By Lemma 15, y must be inside f_{uvw}. W. l. o. g., let y be in f_{uvx}. By the same reasoning, the fat neighbors of y not in $\{u, v, x\}$ must be in the same triangle as y, which due to the connectivity of G^+ and G on fat edges even without the K_3 $G^+[\{u, v, x\}]$ implies that all vertices of G must be in f_{uvx}. However, w is outside f_{uvx}, a contradiction. □

Lemma 17. *No edge $e \in E$ can cross a fat edge $f \in E$ in any RAC drawing $\mathcal{D}(G^+)$ of G^+.*

Proof. Suppose edge $e = \{u, v\} \in E$ crosses the fat edge $f = \{x, y\} \in E$ in $\mathcal{D}(G^+)$. As a fat edge, f shares an edge with a K_3 $\tau = G^+[\{x, y, z\}]$, where $z \in V^+ \setminus V$ is a vertex of a 2-path associated with f. Edge e cannot cross both $\{x, y\}$ and $\{x, z\}$ or both $\{x, y\}$ and $\{y, z\}$ due to forbidden fan-crossings by Lemma 13. Thus, w. l. o. g., u is inside the triangle f_{xyz} and v outside. As there is a path p between u and v which consists only of fat edges and is vertex disjoint with τ, there is an edge of p crossing an edge of triangle f_{xyz}, a contradiction to Lemma 15. □

Lemma 18. *Let $\mathcal{D}(G^+)$ be a RAC drawing of G^+ and let $\mathcal{D}(G)$ be the induced drawing. Then every K_4 is drawn as a kite with planar fat edges in $\mathcal{D}(G)$.*

Proof. By Lemma 16, every K_4 κ is drawn with a right angle crossing. Let $\mathcal{D}(\kappa)$ be the induced drawing of κ. Due to Lemma 13, no other edge of G can cross two (or more) edges of κ. Furthermore, no other vertex of G can be inside $\mathcal{D}(\kappa)$ by the same argument as in the proof of Lemma 16: The pair of crossing edges partitions $\mathcal{D}(\kappa)$ into four triangles. If a vertex is in such a triangle, then all vertices of G must be in the same triangle, since they are connected by fat edges. However, the remaining two vertices of κ cannot be in the triangle. Hence, $\mathcal{D}(\kappa)$ is empty and every K_4 subgraph of G^+ is drawn as kite. By Lemma 17, the crossing edges of the kites cannot be fat. □

As a consequence, in the induced drawing $\mathcal{D}(G)$ every K_3 in G which is no subgraph of a K_4 is drawn as a trivial triangle and every K_4 in G is drawn as kite, i. e., with no further vertex inside. Finally, consider the outer face:

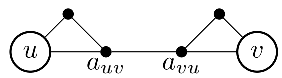

Figure 16: Gadget replacing every edge $\{u, v\}$ in the $\mathcal{NP}$-reduction.

Lemma 19. *Let $\mathcal{D}(G^+)$ be a RAC drawing of G^+ and let $\mathcal{D}(G)$ be the induced drawing. Then the outer face of $\mathcal{D}(G)$ is a trivial triangle.*

Proof. If the outer face of $\mathcal{D}(G)$ is not a trivial triangle, it must be one of the four triangles of a K_4's kite drawing. Let c denote the crossing point of the kite's edges e and f. Then, the interior angle at c must be less than π, which implies a bend at c for both e and f, a contradiction to $\mathcal{D}(G)$ being straight-line. □

So far, we conclude that the induced embedding $\mathcal{E}(G)$ must be as depicted in Fig. 15a if there is a RAC drawing of G^+. However, this embedding is not realizable with right angle crossings.

Lemma 20. *Graph G^+ does not admit a RAC drawing.*

Proof. Assume that G^+ has a RAC drawing $\mathcal{D}(G^+)$. By Lemma 19, the outer face of the induced drawing $\mathcal{D}(G)$ is a trivial triangle τ. Every fat edge of G is a planar edge of a K_4 in $\mathcal{D}(G)$, or more specifically of a kite (Lemma 18). Hence, every edge bounding τ is shared with a kite κ_i for $i = 1, 2, 3$, which is located inside τ. Let c_i denote the point in the plane where the two edges of κ_i cross each other and let τ_i be the non-trivial triangle of the kite embedding of κ_i that is bounded by one of the edges of τ. Then, the interior angle of τ_i at c_i must be $\frac{\pi}{2}$ and subsequently, the two remaining interior angles of τ_i sum up to $\pi - \frac{\pi}{2} = \frac{\pi}{2}$. Observe that the kites' faces are pairwisely disjoint by Lemma 18. Hence, the sum of τ's interior angles must be strictly greater than $\frac{3\pi}{2}$, a contradiction to τ being a triangle. □

7. Recognition

The recognition problem for 1-planar and IC-planar graphs is $\mathcal{NP}$-complete, even if the graphs are 3-connected and are given with a rotation system [6, 15, 27]. However, triangulated, maximal, and optimal 1-planar graphs can be recognized in time $\mathcal{O}(n^3)$ [19], $\mathcal{O}(n^5)$ [12], and $\mathcal{O}(n)$ [14], respectively.

Our $\mathcal{NP}$-hardness result solves an open problem by Zhang [38] and can be obtained from known $\mathcal{NP}$-hardness proofs [15, 27], e. g., by a reduction from 1-planarity as in [15], which replaces every edge $\{u, v\}$ of a graph $G = (V, E)$ by the gadget in Fig. 16. Then, in every IC-planar and even NIC-planar embedding $\mathcal{E}(G')$ of the resulting graph G', every crossed edge must be an edge $\{a_{uv}, a_{vu}\}$ for some $\{u, v\} \in E$ and $\mathcal{E}(G')$ exists if and only if the induced embedding of G is 1-planar. As testing 1-planarity is $\mathcal{NP}$-complete, we obtain:

Corollary 11. *It is $\mathcal{NP}$-complete to test whether a graph is NIC-planar.*

Recently, Brandenburg [13] has used the relationship between 1-planar graphs and hole-free 4-map graphs as in Corollary 2 and the cubic-time recognition algorithm for (hole-free) 4-map graphs of Chen et al. [19] to develop a cubic-time recognition algorithm for triangulated NIC-planar (IC-planar) graphs from which he obtained an $\mathcal{O}(n^5)$ time algorithm for maximal and a cubic-time algorithm for densest NIC-planar (IC-planar) graphs.

8. Conclusion

For a natural subclass of 1-planar graphs, we presented diverging, yet tight upper and lower bounds for maximal graphs. Paralleling the result that there are maximal 1-planar graphs that are sparser than maximal planar graphs, we showed that there are maximal NIC-planar graphs that are sparser than maximal IC-planar graphs. Our tool is a generalized dual graph and a condensation of K_4 subgraphs. Whereas IC-planar graphs are a subset of RAC graphs, we showed that NIC-planar graphs and RAC graphs are incomparable. The proof of Theorem 4 shows, to the best of our knowledge for the first time, that there are non-RAC, 1-planar graphs with a density less than the upper bound for RAC graphs of $4n-10$. Finally, we showed that the recognition of NIC-planar graphs is $\mathcal{NP}$-hard in general, whereas optimal NIC-planar graphs can be recognized in linear time.

Future work are similar characterizations for IC-planar graphs in terms of generalized duals and the linear-time recognition of optimal IC-planar graphs.

References

[1] Ackerman, E., Tardos, G., 2007. On the maximum number of edges in quasi-planar graphs. J. Combin. Theory, Ser. A 114 (3), 563–571.

[2] Alam, M. J., Brandenburg, F. J., Kobourov, S. G., 2013. Straight-line drawings of 3-connected 1-planar graphs. In: Wismath, S., Wolff, A. (Eds.), GD 2013. Vol. 8242 of LNCS. Springer, pp. 83–94.

[3] Albertson, M. O., 2008. Chromatic number, independence ratio, and crossing number. Ars Math. Contemp. 1 (1), 1–6.

[4] Appel, K., Haken, W., 1977. Every planar map is four colorable. part i. discharging. Illinois J. Math. 21, 429–490.

[5] Auer, C., Bachmaier, C., Brandenburg, F. J., Gleißner, A., Hanauer, K., Neuwirth, D., Reislhuber, J., 2016. Outer 1-planar graphs. Algorithmica 74 (4), 1293–1320.

[6] Auer, C., Brandenburg, F. J., Gleißner, A., Reislhuber, J., 2015. 1-planarity of graphs with a rotation system. J. Graph Algorithms Appl. 19 (1), 67–86.

[7] Bachmaier, C., Brandenburg, F. J., Hanauer, K., July 2017. A note on IC-planar graphs. Tech. Rep. arXiv:1707.08652 [cs.DM], Computing Research Repository (CoRR).

[8] Bannister, M. J., Cabello, S., Eppstein, D., 2013. Parameterized complexity of 1-planarity. In: WADS 2013. Vol. 8037 of LNCS. Springer, pp. 97–108.

[9] Barát, J., Tóth, G., September 2015. Improvements on the density of maximal 1-planar graphs. Tech. Rep. arXiv:1509.05548 [math.CO], Computing Research Repository (CoRR).

[10] Bodendiek, R., Schumacher, H., Wagner, K., 1983. Bemerkungen zu einem Sechsfarbenproblem von G. Ringel. Abh. aus dem Math. Seminar der Univ. Hamburg 53, 41–52.

[11] Bodendiek, R., Schumacher, H., Wagner, K., 1984. Über 1-optimale Graphen. Mathematische Nachrichten 117, 323–339.

[12] Brandenburg, F. J., January 2015. On 4-map graphs and 1-planar graphs and their recognition problem. Tech. Rep. arXiv:1509.03447 [cs.CG], Computing Research Repository (CoRR).

[13] Brandenburg, F. J., October 2016. Recognizing IC-planar and NIC-planar graphs. Tech. Rep. arXiv:1610.08884 [cs.DM], Computing Research Repository (CoRR).

[14] Brandenburg, F. J., 2016. Recognizing optimal 1-planar graphs in linear time. Algorithmica. doi:10.1007/s00453-016-0226-8.

[15] Brandenburg, F. J., Didimo, W., Evans, W. S., Kindermann, P., Liotta, G., Montecchiani, F., 2016. Recognizing and drawing IC-planar graphs. Theor. Comput. Sci. 636, 1–16.

[16] Brandenburg, F. J., Eppstein, D., Gleißner, A., Goodrich, M. T., Hanauer, K., Reislhuber, J., 2013. On the density of maximal 1-planar graphs. In: van Kreveld, M., Speckmann, B. (Eds.), GD 2012. Vol. 7704 of LNCS. Springer, pp. 327–338.

[17] Cabello, S., Mohar, B., 2013. Adding one edge to planar graphs makes crossing number and 1-planarity hard. SIAM J. Comput. 42 (5), 1803–1829.

[18] Chen, Z., Grigni, M., Papadimitriou, C. H., 2002. Map graphs. J. ACM 49 (2), 127–138.

[19] Chen, Z., Grigni, M., Papadimitriou, C. H., 2006. Recognizing hole-free 4-map graphs in cubic time. Algorithmica 45 (2), 227–262.

[20] Chiba, N., Yamanouchi, T., Nishizeki, T., 1985. Arboricity and subgraph listing. SIAM J. Comput. 14 (1), 210–223.

[21] Czap, J., Šugarek, P., 2014. Three classes of 1-planar graphs. Tech. Rep. arXiv:1404.1222 [math.CO], Computing Research Repository (CoRR).

[22] Di Battista, G., Eades, P., Tamassia, R., Tollis, I. G., 1999. Graph Drawing: Algorithms for the Visualization of Graphs. Prentice Hall.

[23] Didimo, W., Eades, P., Liotta, G., 2011. Drawing graphs with right angle crossings. Theor. Comput. Sci. 412 (39), 5156–5166.

[24] Dillencourt, M. B., Eppstein, D., Hirschberg, D. D., 2000. Geometric thickness of complete graphs. J. Graph Algorithms Appl. 4 (3), 5–15.

[25] Dolev, D., Leighton, T., Trickey, H., 1984. Planar embedding of planar graphs. Advances in Computing Research 2, 147–161.

[26] Eades, P., Liotta, G., 2013. Right angle crossing graphs and 1-planarity. Discrete Appl. Math. 161 (7-8), 961–969.

[27] Grigoriev, A., Bodlaender, H. L., 2007. Algorithms for graphs embeddable with few crossings per edge. Algorithmica 49 (1), 1–11.

[28] Hong, S., Eades, P., Katoh, N., Liotta, G., Schweitzer, P., Suzuki, Y., 2015. A linear-time algorithm for testing outer-1-planarity. Algorithmica 72 (4), 1033–1054.

[29] Korzhik, V. P., Mohar, B., 2013. Minimal obstructions for 1-immersion and hardness of 1-planarity testing. J. Graph Theor. 72, 30–71.

[30] Král, D., Stacho, L., 2010. Coloring plane graphs with independent crossings. J. Graph Theor. 64 (3), 184–205.

[31] Kyncl, J., 2009. Enumeration of simple complete topological graphs. Eur. J. Comb. 30 (7), 1676–1685.

[32] Liotta, G., 2014. Graph drawing beyond planarity: some results and open problems. In: Bistarelli, S., Formisano, A. (Eds.), Proc. 15th Italian Conference on Theoretical Computer Science. Vol. 1231 of CEUR Workshop Proceedings. CEUR-WS.org, pp. 3–8.

[33] Nash-Williams, C. S. J. A., 1961. Decomposition of finite graphs into forests. J. London Math. Soc. 39 (1), 445–450.

[34] Ringel, G., 1965. Ein Sechsfarbenproblem auf der Kugel. Abh. aus dem Math. Seminar der Univ. Hamburg 29, 107–117.

[35] Schumacher, H., 1986. Zur Struktur 1-planarer Graphen. Mathematische Nachrichten 125, 291–300.

[36] Suzuki, Y., 2010. Optimal 1-planar graphs which triangulate other surfaces. Discrete Math. 310 (1), 6–11.

[37] Suzuki, Y., 2010. Re-embeddings of maximum 1-planar graphs. SIAM J. Discr. Math. 24 (4), 1527–1540.

[38] Zhang, X., 2014. Drawing complete multipartite graphs on the plane with restrictions on crossings. Acta Math. Sinica, English Series 30 (12), 2045–2053.

[39] Zhang, X., Liu, G., 2013. The structure of plane graphs with independent crossings and its application to coloring problems. Central Europ. J. Math 11 (2), 308–321.