The Duals of Upward Planar Graphs on Cylinders*

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Abstract. We consider directed planar graphs with an upward planar drawing on the rolling and standing cylinders. These classes extend the upward planar graphs in the plane. Here, we address the dual graphs. Our main result is a combinatorial characterization of these sets of upward planar graphs. It basically shows that the roles of the standing and the rolling cylinders are interchanged for their duals.

1 Introduction

Directed graphs are used as a model for structural relations where the edges express dependencies. Such graphs are often acyclic and are drawn as hierarchies using the framework introduced by Sugiyama et al. [21]. This drawing style transforms the edge direction into a geometric direction: all edges point upward. If only plane drawings are allowed, one obtains *upward planar graphs*, for short **UP**. These graphs can be drawn in the plane such that the edge curves are monotonically increasing in y-direction and do not cross. Hence, **UP** graphs respect the unidirectional flow of information as well as planarity.

There are some fundamental differences between upward planar and undirected planar graphs. For instance, there are several linear time planarity tests [17], whereas the recognition problem for **UP** is $\mathcal{NP}$-complete [13]. The difference between planarity and upward planarity becomes even more apparent when different types of surfaces are studied: For instance, it is known that every graph embeddable on the plane is also embeddable on any surface of genus 0, e. g., the sphere and the cylinder, and vice versa. However, there are graphs with an upward embedding on the sphere with edge curves increasing from the south to the north pole, which are not upward planar [16]. The situation becomes even more challenging if upward embeddability is extended to other surfaces even if these are of genus 0.

Upward planarity on surfaces other than the plane generally considers embeddings of graphs on a fixed surface in $\mathbb{R}^3$ such that the curves of the edges are monotonically increasing in y-direction. Examples for such surfaces are the standing [7, 14, 19, 20, 22] and the rolling cylinder [7], the sphere and the truncated sphere [10,12,15,16], and the lying and standing tori [9,11]. We generalized

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upward planarity to arbitrary two-dimensional manifolds endowed with a vector field which prescribes the direction of the edges [2]. We also studied upward planarity on standing and rolling cylinders, where the former plays an important role for radial drawings [3] and the latter in the context of recurrent hierarchies [4]. We showed that upward planar drawings on the rolling cylinder can be simplified to polyline drawings, where each edge needs only finitely many bends and at most one winding around the cylinder [7]. The same holds for the standing cylinder, where all windings can be eliminated [7]. In accordance to [2], we use the fundamental polygon to define the plane, the standing and the rolling cylinders. The *plane* is identified with $I \times I$, where I is the open interval from -1 to $+1$, i. e., $I \times I$ is the (interior of the) square with side length two. The *rolling* (*standing*) *cylinder* is obtained by identifying the bottom and top (left and right) sides. By identifying the boundaries of I, we obtain $I_\circ$. Then, the standing and the rolling cylinder are defined by $I_\circ \times I$ and $I \times I_\circ$, respectively. Let **RUP** be the set of graphs which can be drawn on the rolling cylinder such that the edge curves do not cross and are monotonically increasing in y-direction. If the edge curves are permitted to be non-decreasing in y-direction, horizontal lines are allowed. Since the top and bottom sides of the fundamental polygon are identified, "upward" means that edge curves wind around the cylinder all in the same direction. Specifically, **RUP** allows for cycles. Accordingly, let **SUP** denote the class of graphs with a planar drawing on the standing cylinder and increasing curves for the edges and let **wSUP** be the corresponding class of graphs with non-decreasing curves. The novelty of **wSUP** graphs are cycles with horizontal curves, whereas **SUP** graphs are acyclic, i. e., $\mathbf{SUP} \subsetneq \mathbf{wSUP}$. In [2] we established that a graph is in **SUP** if and only if it is upward planar on the sphere. These *spherical* graphs were studied in [10,12,15,16]. Finally, let **UP** be the class of upward planar graphs (in the plane) [8,18]. Note that for **UP** and **RUP** graphs non-decreasing curves can be replaced by increasing ones and the corresponding classes coincide [2].

Upward planar graphs in the plane and on the sphere or on the standing cylinder were characterized by using *acyclic dipoles*. An acyclic dipole is a directed acyclic graph with a single source s and a single sink t. More specifically, a graph G is **SUP**/spherical if and only if it is a spanning subgraph of a planar acyclic dipole [14,16,19]. The idea behind acyclic dipoles is that s corresponds to the south and t to the north pole of the sphere. Moreover, a graph G is in **UP** if and only if the dipole has in addition the (s, t) edge [8,18].

In contrast, there is no related characterization of **RUP** graphs. Acyclic dipoles cannot be used since **RUP** graphs may have cycles winding around the rolling cylinder. However, the idea behind dipoles can be applied indirectly to **RUP** graphs, namely, to their duals. For this, we generalize acyclic dipoles to *dipoles* which may also contain cycles.

Section 2 provides the necessary definitions. We develop our new characterization of **RUP** and **SUP** graphs in terms of their duals in Sect. 3. In Sect. 4 we obtain related results for **wSUP** graphs. All formal proofs can be found in [1].

2 Preliminaries

The graphs in this work are connected, planar (unless stated otherwise), directed multigraphs $G = (V, E)$ with non-empty sets of vertices V and edges E, where pairs of vertices may be connected by multiple edges. G can be drawn in the plane such that the vertices are mapped to distinct points and the edges to non-intersecting Jordan curves. Then, G has a planar *drawing*. It implies an *embedding* of G, which defines (cyclic) orderings of incident edges at the vertices. In the following, we only deal with embedded graphs and all paths and cycles are simple.

A *face* f of G is defined by a (underlying undirected) circle $C = (v_1, e_1, v_2, e_2, \dots, v_{k-1}, e_{k-1}, v_k = v_1)$ such that $e_i \in E$ is the direct successor of $e_{i-1} \in E$ according to the cyclic ordering at v_i. The edges/vertices of C are said to be the *boundary* of f and C is a *clockwise traversal* of f. Accordingly, the *counterclockwise traversal* of f is obtained by choosing the predecessor edge at each vertex in the circle. The embedding defines a unique *(directed) dual* graph $G^* = (F, E^*)$, whose vertex set is the set of faces F of G [5]. Let $f \in F$ be a face of G and $e = (u, v) \in E$ be part of its boundary. If the counterclockwise traversal of f passes e in its direction, we say that *f is to the left of e*. If the same holds for e and another face g in clockwise direction, then *g is to the right of e*. For each edge $e \in E$ there is an edge in E^* from the face to the left of e to the face right of e. This definition establishes a bijection between E and E^*. Whenever necessary, we refer to G as the *primal* of G^*. By vertex we mean an element of V, whereas the vertices F of G^* are called faces.

Note that G^* is connected and the dual of G^* is isomorphic to the *converse* G^{-1} of G where all edges are reversed, since G is connected. Hence, an embedding of G implies an embedding of G^*, and vice versa. G and G^{-1} share many properties, see Proposition 1.

An embedding of a graph is an X *embedding* with $X \in \{\mathbf{RUP}, \mathbf{SUP}, \mathbf{wSUP}, \mathbf{UP}\}$ if it is obtained from an X drawing. For every graph in class X, we assume that a corresponding X embedding is given. Given an embedded graph G, a face *f is to the left of a face g* if there is a path $f \rightsquigarrow g$ in G^*. Note that a face can simultaneously lie to the left and to the right of another face, and "to the left" does not directly correspond to the geometric left-to-right relation in a drawing. A cycle in a **RUP** embedding winds exactly once around the cylinder [7]. We say that a face $f \in F$ *lies left (right) of a cycle C* if there is another face $g \in F$ such that f is to the left (right) of g and each path $f \rightsquigarrow g$ in the dual contains at least one edge of C. Each edge/face of f's boundary is then also said to *lie to the left (right) of C*.

Next we introduce graphs which represent the high-level structure of a given graph and which inherit its embedding. Let the equivalence class $[v]$ denote the set of vertices of the strongly connected component containing the vertex $v \in V$ and let $\mathbb{V}$ be the set of strongly connected components of G. The *component graph* $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ of G contains an edge $([v], [w]) \in \mathbb{E}$ for each original edge $(v, w) \in E$ with $[v] \neq [w]$. $\mathbb{G}$ is an acyclic multigraph which inherits the embedding of G. A component $\gamma \in \mathbb{V}$ is a *compound*, if it contains more than one vertex or consists of a single vertex with a loop. Its induced subgraph is denoted by $G_\gamma \subseteq G$. For the sake of convenience, we identify G_γ with γ and call both compound. The set of all compounds is denoted by $\mathbb{V}_C$. Each component $[v]$ that is *not a compound* consists of a single vertex v and is called *trivial component*. A trivial component which is a source (sink) in $\mathbb{G}$ is called *source (sink) terminal* and the set of all terminals is denoted by $\mathbb{T} \subseteq \mathbb{V}$. Based on the component graph, we define the *compound graph* $\overline{\mathbb{G}} = (\mathbb{V}_C \cup \mathbb{T}, \overline{\mathbb{E}})$, whose vertices are the compounds and terminals. Let $u, v \in \mathbb{V}_C \cup \mathbb{T}$ be two vertices of the compound graph. There is an edge $(u, v) \in \overline{\mathbb{E}}$ if there is a path $u \rightsquigarrow v$ in $\mathbb{G}$ which internally visits only trivial components. Note that $\overline{\mathbb{G}}$ is a simple graph. Each edge $\tau \in \overline{\mathbb{E}}$ corresponds to a set of paths in $\mathbb{G}$. Denote by $\mathbb{G}_\tau$ the subgraph of $\mathbb{G}$ which is induced by the set of paths belonging to edge τ. We call τ and its induced graph $\mathbb{G}_\tau$ *transit*. See Fig. 1 for an example, where the fundamental polygon of the rolling cylinder is represented by rectangles with identified bottom and top sides. Based on these definitions, we are now able to define dipoles.

Definition 1. *A graph is a* dipole *if it has exactly one source s and one sink t and its compound graph is a path from s to t.*

Note that similar to the definition of *st*-graphs [8, 18], a dipole is not necessarily planar.

Lemma 1. *Let $G = (V, E)$ be a graph with a source s and a sink t. Then, G is a dipole if and only if every path $s \rightsquigarrow t$ contains at least one vertex of each compound and for every vertex $v \in V$ there are paths $s \rightsquigarrow v$ and $v \rightsquigarrow t$.*

Proposition 1. *A graph G is (i) acyclic, (ii) strongly connected, (iii) upward planar, or (iv) a dipole if and only if the same holds for its converse G^{-1}.*

Thus, in the subsequent statements on the relationship between a graph G and its dual G^*, the roles of G and G^* are interchangeable.

Lemma 2. *A graph G is acyclic if and only if its dual G^* is strongly connected.*

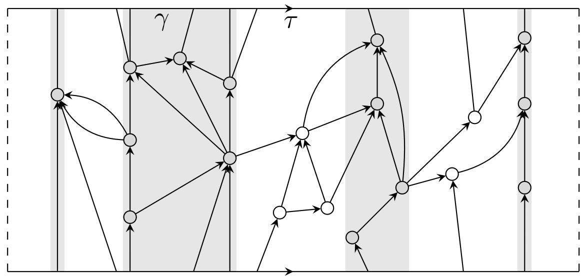
The proof is deduced from the one for polynomial solvability of the feedback arc set problem on planar graphs as given in [5].

3 RUP and SUP Graphs and their Duals

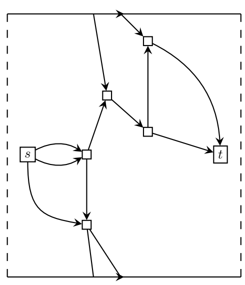
We consider **RUP** graphs, i. e., upward planar graphs on the rolling cylinder, and characterize them in terms of their duals. Our main result is:

Theorem 1. *A graph G is a **RUP** graph if and only if G is a spanning subgraph of a planar graph H without sources or sinks whose dual H^* is a dipole.*

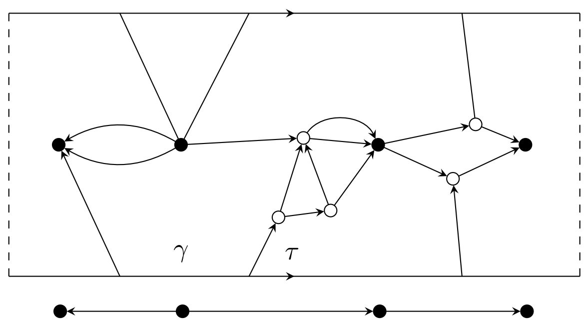
The theorem is proved by a series of lemmata which are also of interest in their own. For our first observation, consider the **RUP** drawing of graph G in Fig. 1(a), where all vertices within a compound are drawn on a shaded background. The



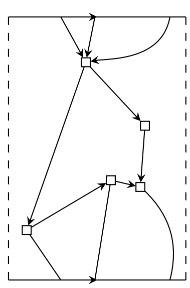
(a) Graph $G \in$ **RUP**



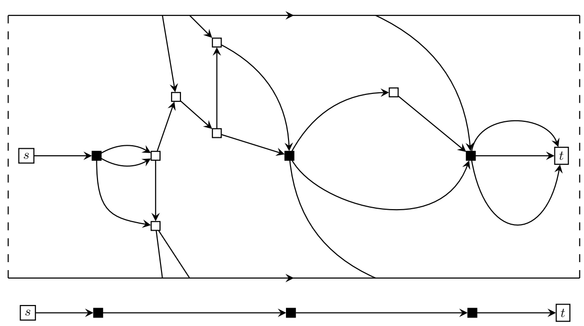
(b) The dual of the second compound γ of G



(c) The component graph $\mathbb{G}$ and the compound graph $\overline{\mathbb{G}}$ of G



(d) The dual of the second transit τ of G



(e) The component graph $\mathbb{G}^*$ and the compound graph $\overline{\mathbb{G}^*}$ of the dual G^* with $s, t \in \mathbb{T}$

Fig. 1. A **RUP** example

component graph $\mathbb{G}$ of G is displayed in Fig. 1(c) along with its compound graph $\overline{\mathbb{G}}$ below, where the compounds are shaded black. Note that $\overline{\mathbb{G}}$ has the structure of an (undirected) path. Due to Lemma 2, each transit, i. e., edge in $\overline{\mathbb{G}}$, becomes a compound and each compound, i. e., vertex in $\overline{\mathbb{G}}$, becomes a transit in the dual G^* of G. Hence, the path-like structure of $\overline{\mathbb{G}}$ must carry over to the compound graph $\overline{\mathbb{G}^*}$ of G^*. Moreover, since all cycles in the **RUP** drawing have the same orientation, i. e., they all wind around the cylinder in the same direction, the transits in G^* point into the same direction. Also note that G contains neither sources nor sinks, i. e., both the left and right border of the drawing are directed cycles C_l and C_r, respectively. Hence, in the dual G^* of G, the face to the left of C_l is a source s and the face to the right of C_r is a sink t. All these observations together indicate that the compound graph of G^* is a path $s \rightsquigarrow t$, i. e., G^* is a dipole. Indeed, this can be seen for the example in Fig. 1(e), where the component graph of G^* and its compound graph are depicted.

Lemma 3. *The dual G^* of a **RUP** graph G without sources and sinks is a dipole.*

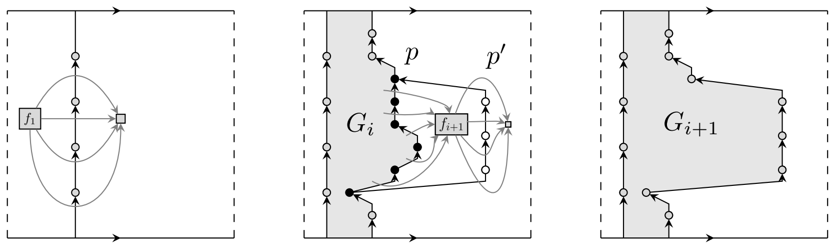
For the following lemma, there is a physical interpretation: Consider an upward drawing of a planar acyclic dipole on the standing cylinder and suppose that an electric current flows from the bottom to the top of the cylinder in direction of the edges. This current induces a magnetic field wrapping around the standing cylinder. Intuitively, by Lemma 4, we can show that a dipole's dual is upward planar with respect to the induced magnetic field, i. e., its embedding is a **RUP** embedding.

Lemma 4. *The embedding of a strongly connected graph G is a **RUP** embedding if and only if its dual G^* is an acyclic dipole.*

The only-if direction follows from Lemmata 2 and 3. For the if direction, we give a sketch of the proof. Let $G^* = (F, E^*)$ be the dual of G and consider a topological ordering $f_1, \ldots, f_k$ of the faces F. We subsequently process the faces according to their topological ordering and construct a drawing of G by placing the edges and vertices of the faces in that order. We start with the only source f_1 in G^*, which corresponds to a directed cycle in G (Fig. 2(a)). In the drawing with the boundaries of all faces $f_1, \ldots, f_i$, we can show that one part of the boundary of face f_{i+1} consists of a directed path $p = (u, \ldots, v)$ along the right border of the current drawing (solid black vertices in Fig. 2(b)). The second part of the boundary, absent in the current drawing, is also a directed path $p' = (u, \ldots, v)$ with the same end points and direction as p (white vertices in Fig. 2(b)). The drawing can be augmented by this path p' while preserving planarity and all edges are monotonically increasing in y-direction (Fig. 2(c)).

Since each **SUP** graph is a subgraph of a planar, acyclic dipole [16], Lemma 4 implies:

Corollary 1. *The dual G^* of a strongly connected **RUP** graph G is in **SUP**.*



(a) Base case (b) Induction step (c) After the induction step

Fig. 2. Inductive construction of a **RUP** drawing from its dual

Consider again the component graph $\mathbb{G}$ and its compound graph $\overline{\mathbb{G}}$ in Fig. 1(c) of the **RUP** graph G in Fig. 1(a). In the dual G^* of G, compounds and transits of G swap their roles, i. e., compounds become transits and vice versa, cf. Fig. 1(e). As a compound of G is a strongly connected **RUP** graph, its dual is an acyclic dipole by Lemma 4. For instance, consider the second compound γ in Fig. 1(a), i. e., the vertices on the second shaded area labeled with γ. Its dual is indeed an acyclic dipole as depicted in Fig. 1(b). For the transits, the same holds but with swapped roles, i. e., the dual of a transit is a strongly connected **RUP** graph. As an example, the dual of the second transit τ in Fig. 1(a) is shown in Fig. 1(d) and it is indeed a strongly connected **RUP** graph. The following lemma subsumes these observations.

Lemma 5. *Let G be a **RUP** graph without sources and sinks and $\overline{\mathbb{G}} = (\mathbb{V}_C, \overline{\mathbb{E}})$ be its compound graph. Then,*

*(i) the dual of each compound $\gamma \in \mathbb{V}_C$ is a planar, acyclic dipole and, thus, it is in **SUP**.*
*(ii) each transit $\tau \in \overline{\mathbb{E}}$ is a planar, acyclic dipole and, thus, its dual is a strongly connected **RUP** graph.*

Both *(i)* and *(ii)* follow from Lemma 4. For *(ii)* note that the graph induced by a transit is an acyclic dipole.

By Lemma 3 we have seen that the dual of a **RUP** graph that contains neither sources nor sinks is a dipole. Also the converse holds:

Lemma 6. *A graph G without sources and sinks is a **RUP** graph if its dual G^* is a dipole.*

Consider again the example **RUP** graph in Fig. 1(a) and the compound graph $\overline{\mathbb{G}^*}$ of its dual G^*. Since G^* is a dipole, $\overline{\mathbb{G}^*}$ is a path $p = (s, \tau_1^*, \gamma_1^*, \tau_2^*, \gamma_2^*, \ldots, \tau_4^*, t)$ consisting of compounds γ_i^*, transits τ_j^*, and two terminals s and t. Note that each element on p corresponds to a subgraph in the primal G, i. e., for each

γ_i^* there is a transit τ_i in G and for each τ_j^* there is a compound γ_j in G. In the proof of Lemma 6, we construct a **RUP** drawing of G by subsequently processing the elements of p. We start with transit τ_1^*, whose induced subgraph in G^* is an acyclic dipole, and obtain a **RUP** drawing of γ_1 which respects the given embedding by Lemma 4. Then we proceed with γ_1^*, a compound in G^*, for which we obtain a **SUP** drawing of τ_1 which respects the given embedding by Lemma 4. However, this **SUP** drawing is upward only with respect to the x-direction, i. e., from left to right. We transform this drawing, while preserving its embedding, such that it is also upward in y-direction. The so obtained drawing of τ_1 is then attached to the right border of the drawing of γ_1. Then, the drawing of γ_2 is attached to the right side of τ_1 and so forth until we reach t. Note that since all transits τ_j^* point into the same direction in $\overline{\mathbb{G}^*}$, i. e., from s to t, all cycles of the compounds in G have the same orientation in the obtained drawing, i. e., they all wind around the cylinder in the same direction.

Lemmata 3 and 6 both require that the graph at hand contains neither sources nor sinks. At a first glance, this requirement seems to be a strong limitation. However, in the following lemma we show that each **RUP** graph can be augmented by edges such that all sources and sinks vanish while still preserving **RUP** embeddability.

Lemma 7. *A **RUP** graph G is a spanning subgraph of a **RUP** graph H without sources and sinks.*

The proof shows that each source (sink) can be connected to another vertex while preserving the upward planar drawability. We follow the construction of the proof of Theorem 1 in [16], which shows that every graph in **SUP** is a spanning subgraph of a planar dipole. Alternatively, the proof can be obtained using techniques from [7].

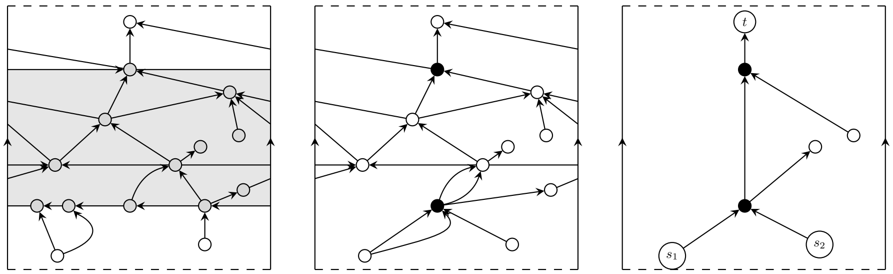
The proof of Theorem 1 is now complete. The only-if direction follows from Lemmata 7 and 3 and the if direction is a consequence of Lemma 6 and the fact that every subgraph of a **RUP** graph is a **RUP** graph.

4 wSUP Graphs and their Duals

We now turn to spherical graphs and upward planar embeddings on the standing cylinder. These graphs were characterized as spanning subgraphs of planar, acyclic dipoles [14,16,19]. We already provided a new characterization for **SUP** in terms of dual graphs in Lemma 4 in combination with Proposition 1. Now we consider graphs which have a weakly upward planar drawing on the standing cylinder. These graphs have not been characterized before.

For a start, consider an upward drawing of a **wSUP** graph. If there are cycles, they must wind around the cylinder horizontally, which leads us to the following observation.

Lemma 8. *Let G be a graph in **wSUP**. Then, all cycles of G are (vertex) disjoint.*



(a) Graph $G \in$ **wSUP** (b) The component graph $\mathbb{G}$ of G (c) The compound graph $\overline{\mathbb{G}}$ of G

Fig. 3. A **wSUP** example

For the characterization of **wSUP** graphs, we use supergraphs which may have an extra source or sink and extend techniques for **SUP** graphs from [16].

Lemma 9. *A graph G is a **wSUP** graph if and only if it has a **wSUP** supergraph $H \supseteq G$ with one source and one sink.*

Consider again an upward drawing of a **wSUP** graph G, e. g., the one depicted in Fig. 3. The cycles subdivide the graph into *sections* as in Fig. 3(a), where the intermediate section is shaded gray. In the component graph, cycles are merged into non-trivial strongly connected components (Fig. 3(b)). The corresponding compound graph has a structure as in Fig. 3(c). We proceed section-wise and eliminate sources and sinks as in [16] except for one source in the lowermost section and one sink in the uppermost one, where the lowermost section is not limited by a cycle from below and the uppermost section by a cycle from above. If any of these two sections is empty, a new source or sink is added to the section and connected to the cycle above or below, respectively. This leaves us with a **wSUP** graph with exactly one source and one sink. Conversely, any subgraph of a **wSUP** graph is in **wSUP**.

We are now able to give a first characterization of **wSUP** graphs.

Theorem 2. *A graph G is a **wSUP** graph if and only if it has a supergraph $H \supseteq G$ such that H is a planar dipole whose cycles are (vertex) disjoint.*

The supergraph H of G can be constructed according to Lemma 9 and is a dipole. By Lemma 8, H has only disjoint cycles. We can obtain a **wSUP** drawing for a planar dipole H whose cycles are disjoint by partitioning the dipole into its compounds and transits. Since transits are acyclic dipoles, the induced subgraph can be drawn upward according to its **SUP** embedding. The compounds consist of single cycles only, which we draw horizontally, i. e., such that each winds around the cylinder once. So H has a **wSUP** drawing and the implied embedding is a **wSUP** embedding. Since G is a subgraph of H, also G is in **wSUP**.

Next, we turn to the duals of **wSUP** graphs. Recall from Lemma 4 in combination with Proposition 1 that a graph with one source and one sink is in **SUP** if and only if its dual is a strongly connected **RUP** graph. Introducing vertex disjoint cycles, the characterization via dual graphs now reads as follows.

Theorem 3. *A graph G with exactly one source and sink is a* ***wSUP*** *graph if and only if its dual G^* is a* ***RUP*** *graph that has no trivial strongly connected components.*

We know from Theorem 1 and Proposition 1 that G^* is in **RUP** since G is a dipole. If G is acyclic, G^* is strongly connected by Lemma 4 and Proposition 1. Otherwise, G consists of compounds and transits. The duals of the transits are strongly connected **RUP** components with at least one edge and, therefore, not trivial. Now consider a compound in G. It consists of a single cycle C, which implies that its dual consists of simple edges from the faces to the left of C to the faces to its right, which themselves are also part of the strongly connected **RUP** components. Hence, all vertices are contained in compounds and G^* has no trivial strongly connected components.

Conversely, a similar argument shows that only single, disjoint cycles can occur in the primal graph of a **RUP** graph without any trivial strongly connected components. Furthermore, by Lemma 3 and Proposition 1, G is a dipole and, therefore, has only one source and one sink. By Theorem 2, G is in **wSUP**.

From Theorem 3 and Lemma 9 we directly obtain the following corollary, which concludes our second characterization of **wSUP** graphs.

Corollary 2. *Every* ***wSUP*** *graph G has a* ***wSUP*** *supergraph H whose dual H^* is a* ***RUP*** *graph without trivial strongly connected components.*

5 Summary

We have shown that a directed graph has a planar upward drawing on the rolling cylinder if and only if it is a spanning subgraph of a planar graph without sources and sinks whose dual is a dipole. This result completes the known characterizations of planar upward drawings in the plane [8, 18] and on the sphere [10, 12, 15, 16]. Every **SUP** graph is a spanning subgraph of a planar, acyclic dipole and every **UP** graph is a spanning subgraph of a planar, acyclic dipole with an *st*-edge. Moreover, a graph has a weakly upward drawing on the standing cylinder if and only if it is a subgraph of a planar dipole with disjoint cycles.

Concerning dual graphs, the duals of the acyclic components of **RUP** graphs are in **RUP** and the duals of the strongly connected components are in **SUP**. In particular, the dual of a strongly connected **RUP** graph is in **SUP**. Every **wSUP** graph has a planar supergraph whose dual is a **RUP** graph without trivial strongly connected components.

Future work is to investigate whether the characterization by means of dual graphs leads to new insights on the upward embeddability on other surfaces, e. g., the torus. Also, the duals of quasi-upward planar graphs [6] shall be considered.

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